Verifying Concurrent Graph Algorithms

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22 November 2016

Concurrent Program Logic Genealogy

Verifying **concurrent** algorithms is difficult…

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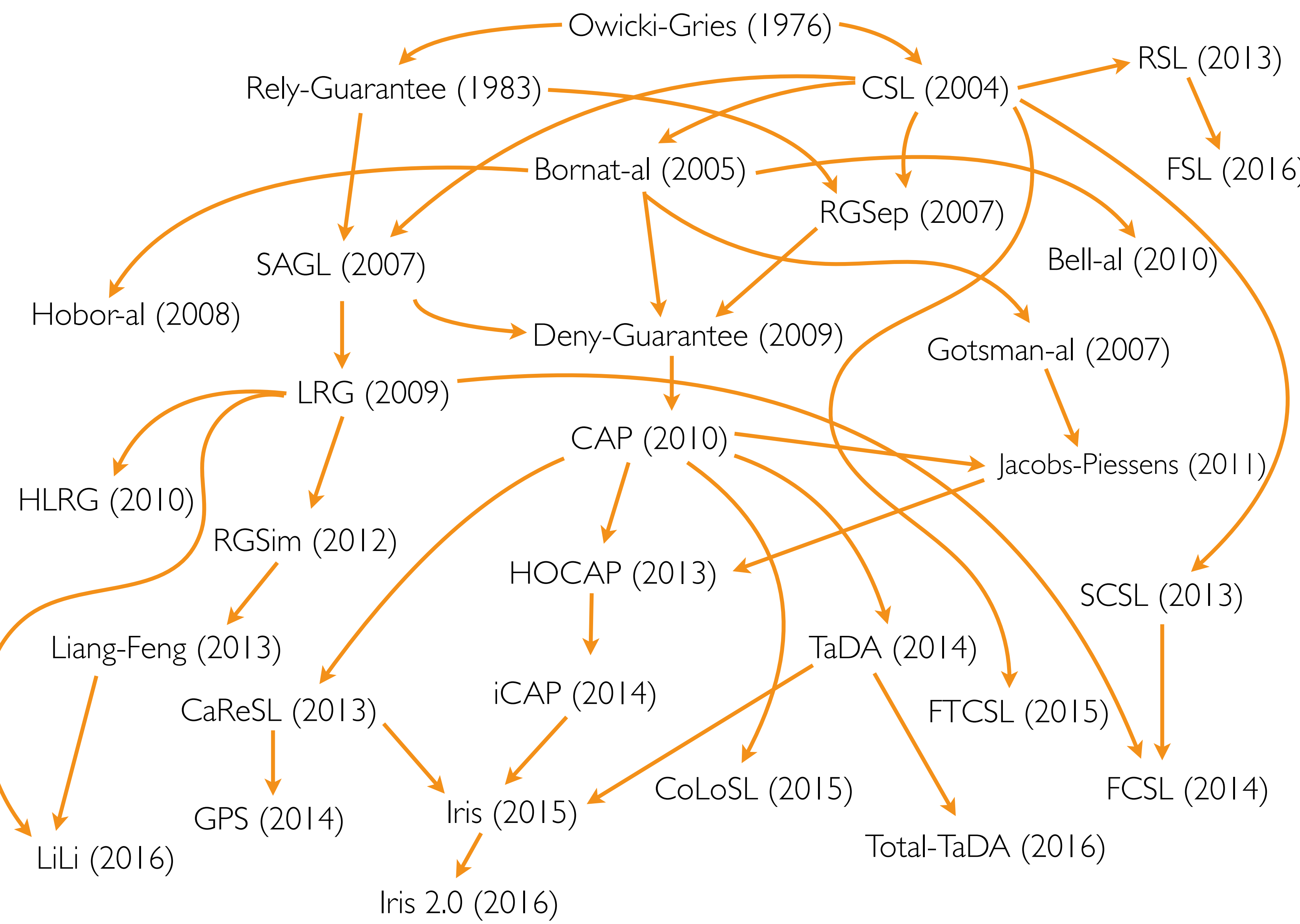

Graph credit: Ilya Sergey

Graph Algorithms

Verifying **graph** algorithms is difficult…

- Subtle correctness argument
- Overlapping structure (unspecified sharing via pointer aliasing)
  - Non-compositional reasoning (preventing the use of the frame rule)

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Local reasoning, separation and aliasing

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Abstract

Structures built by pointer aliasing, such as DAGs and graphs, are notoriously tricky to deal with. The mechanisms of separation logic can deal with these structures, but so far this has been done by the maintenance of a global invariant. Specifications and proofs which use local reasoning, and which may point the way to a structured programming for pointers, are discussed. An idiom for inclusion where one structure is included in another, is presented. A ... graphs' – graphs with dangling pointers – is used

Pointers have been thought of as the gotos of data structuring, and so they can be, but if we can organise things so that our data structures and our programs come apart neatly in similar ways – structured reasoning for the heap – we shall perhaps be able to use pointers reliably. Separation logic facilitates local reasoning about parts of data structures: if an assignment or a procedure call alters only one corner of one data structure, then we can easily express and exploit the fact that it doesn't alter any other part of it nor any part of any separate data structure. The advantages can be seen most clearly in Yang's proof of the Schorr-Waite algorithm [12, 13], which elegantly avoids the struggles of [1] to show the same result using Hoare logic and array assignment. Apart from that work and Torp-Smith's proof of the Cheney algorithm [11], most work using separation logic has focussed on lists and trees, where separation is easily described.

In this paper we discuss three formal proofs (alas, not yet mechanically checked) of algorithms which deal with internal sharing of structure; two of the proofs deal with aliasing as well. Our aim has been to make the reasoning as local as possible. Although we can see considerable possibilities for further improvement, we can already claim significant success in extending the range of separation logic to new and difficult areas.

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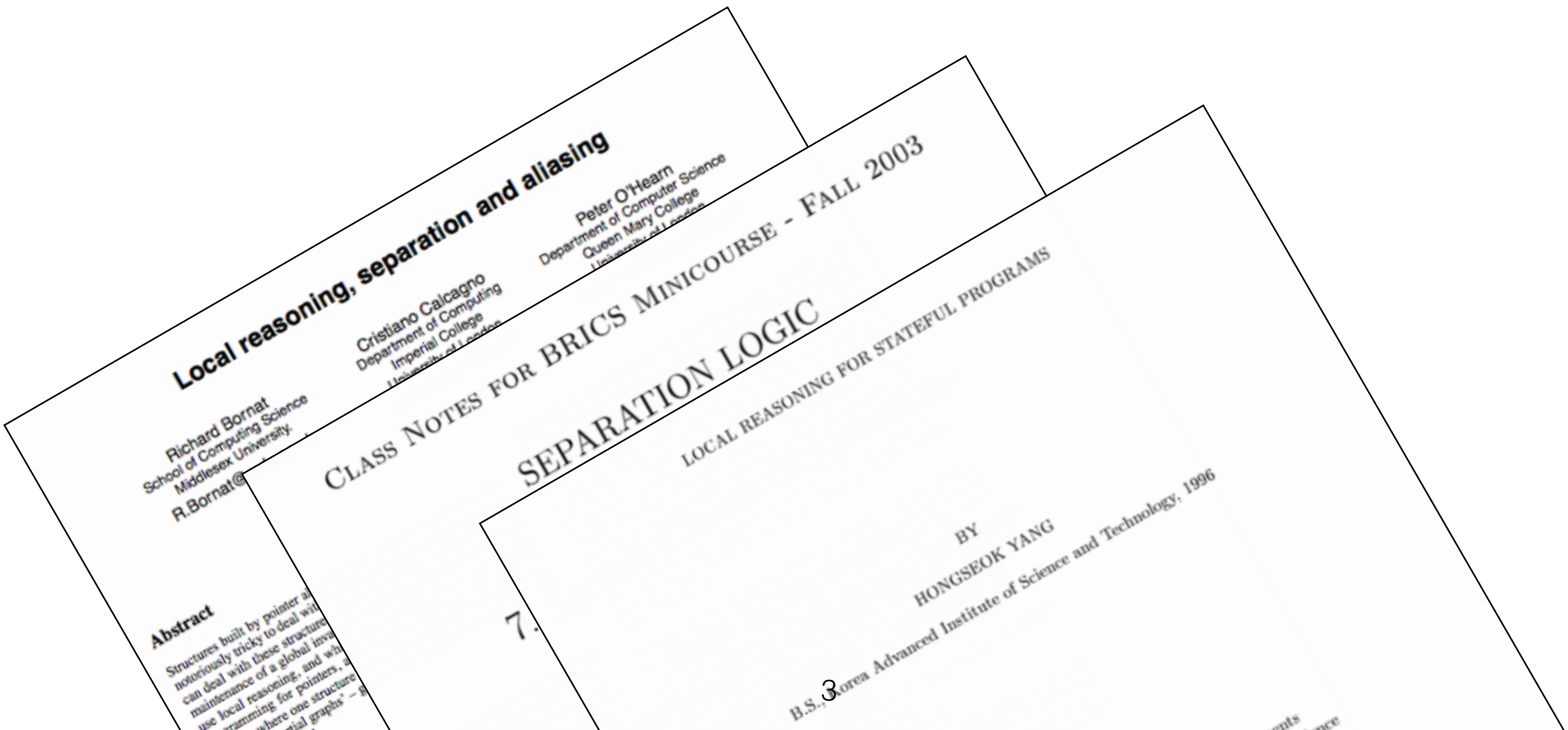

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Concurrent Graph Algorithms

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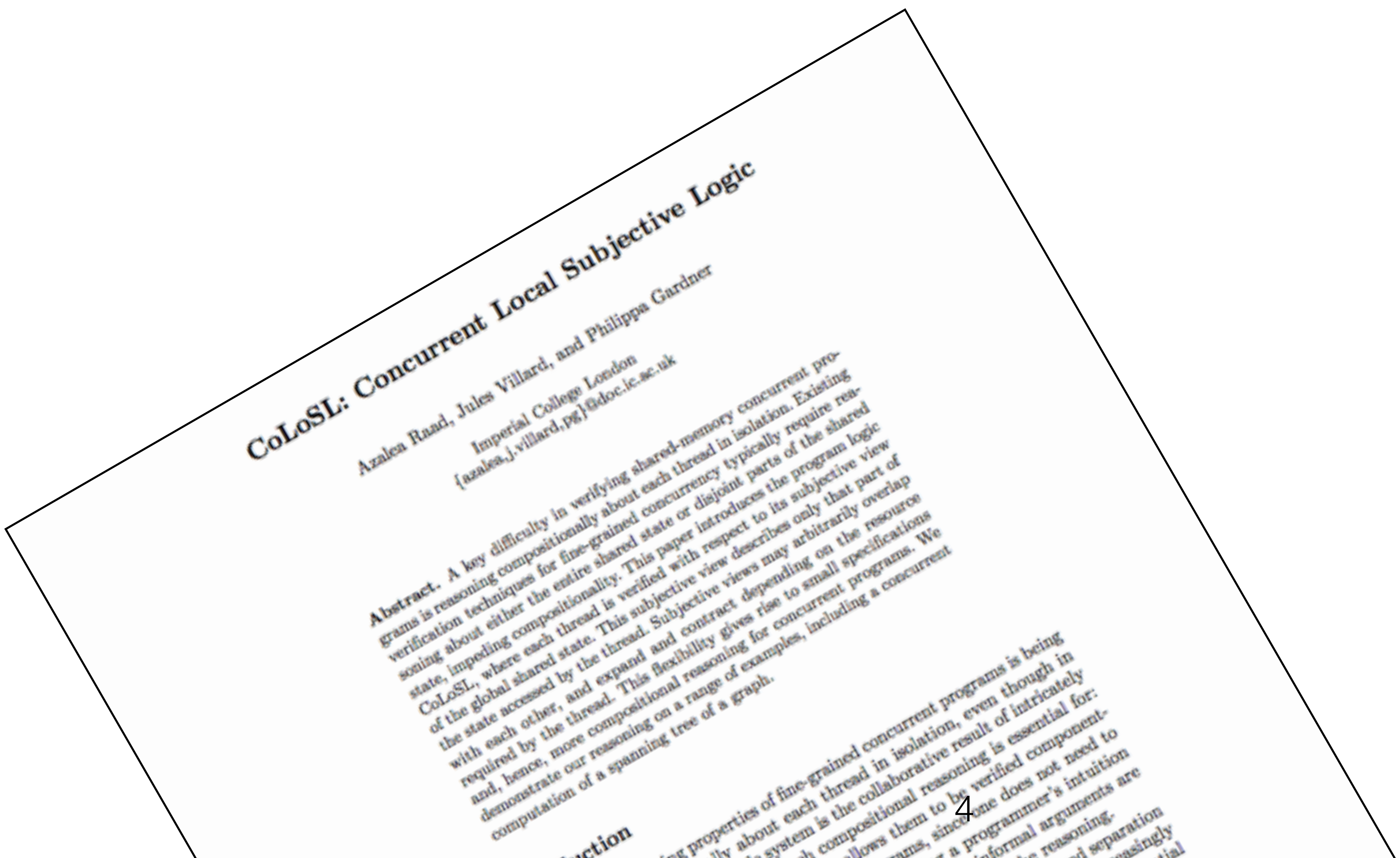

CoLoSL: Concurrent Local Subjective Logic

Azalea Raad, Jules Villard, and Philippa Gardner

Imperial College London
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Abstract. A key difficulty in verifying shared-memory concurrent programs is reasoning compositionally about each thread in isolation. Existing verification techniques for fine-grained concurrency typically require reasoning about either the entire shared state or disjoint parts of the shared state, impeding compositionality. This paper introduces the program logic CoLoSL, where each thread is verified with respect to its subjective view of the global shared state. This subjective view describes only that part of the state accessed by the thread. Subjective views may arbitrarily overlap with each other, and expand and contract depending on the resource required by the thread. This flexibility gives rise to small specifications and, hence, more compositional reasoning for concurrent programs. We demonstrate our reasoning on a range of examples, including a concurrent computation of a spanning tree of a graph.

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Contributions

- Verified 4 concurrent fine-grained graph algorithms
  - Copying dags (directed acyclic graphs)
  - Speculative variant of Dijkstra's shortest path
  - Computing the spanning tree of a graph
  - Marking a graph

Contributions

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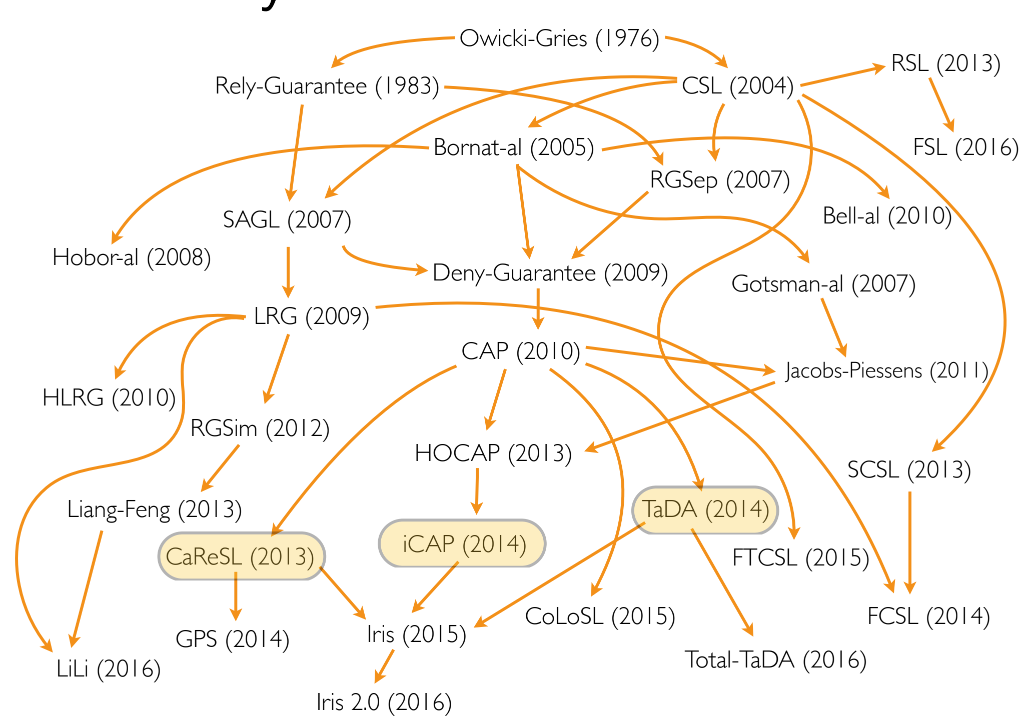

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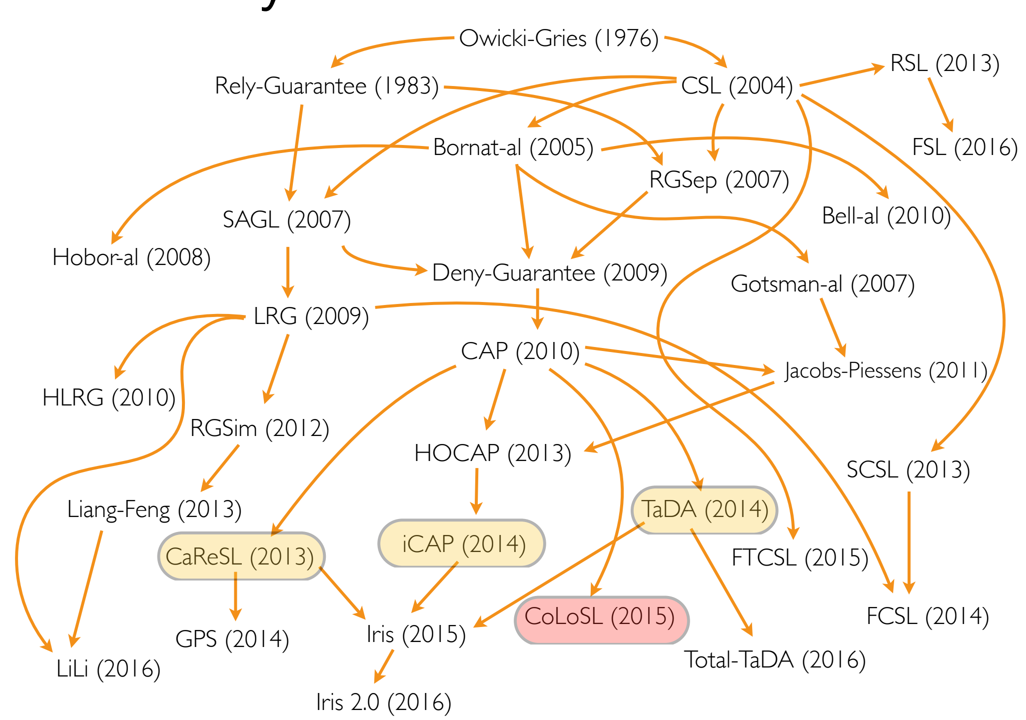

This Talk

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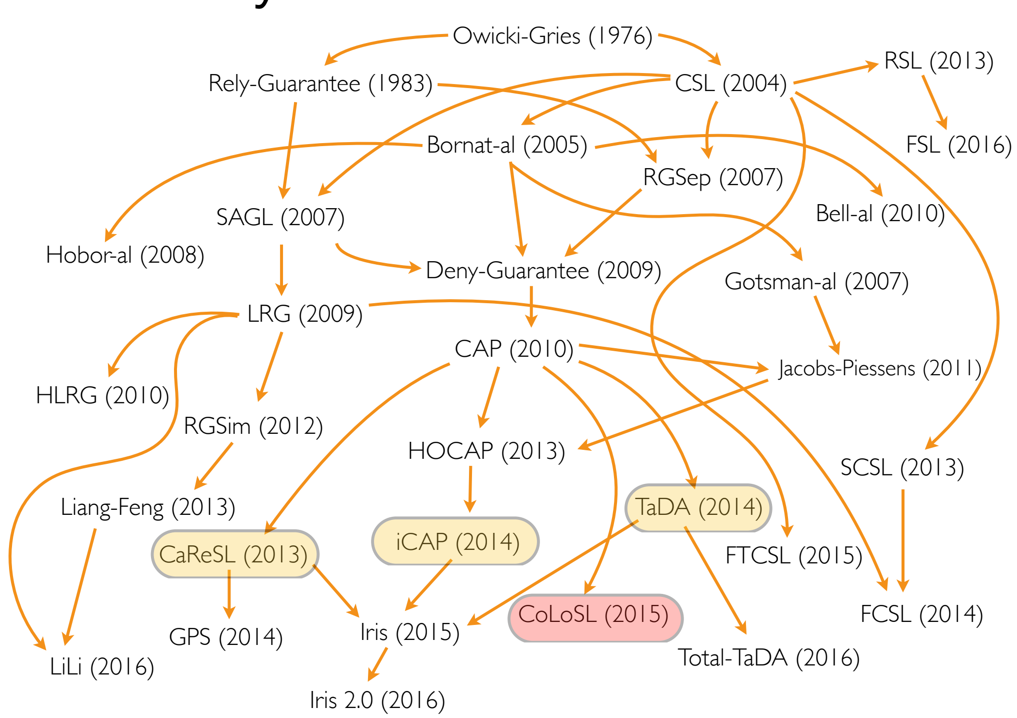

Copying Binary DAGs

```
struct node {struct node *c, *l, *r}
copy_dag(struct node *x) {
  struct node *l, *r, *ll, *rr, *x';  bool b;
  if (!x) {return 0;}
  x' = malloc(sizeof(struct node));
  b = <CAS(x->c, 0, x')>;
  if (b) {
    l = x->l; r = x->r;
    ll = copy_dag(l)  ||  rr = copy_dag(r)
    <x'->l = ll>; <x'->r = rr>;
    return x';
  } else {
    free(x', sizeof(struct node));  return x->c;
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atomic blocks

`copy_dag(x)` Specification

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  }
}
```

- Specification challenges
  - When **`copy_dag(x)`** returns, **`x`** is copied but its children may not be
  - If **`x`** is already copied, **`copy_dag(x)`** simply returns:

    the thread that copied **`x`** has made a <u>*promise*</u> to visit **`x`**'s children and ensure they are copied

`copy_dag(x)`: A Trace

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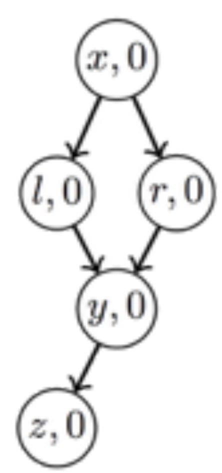

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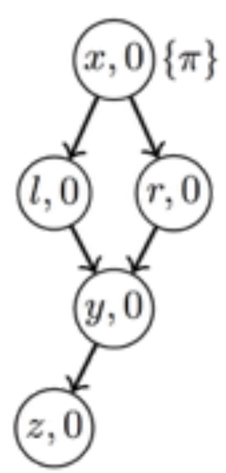

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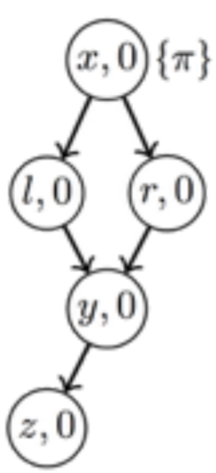

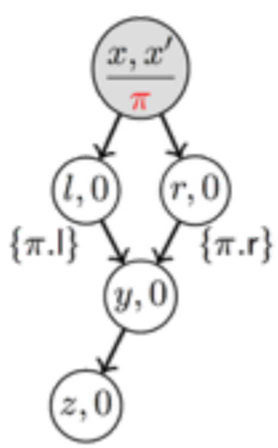

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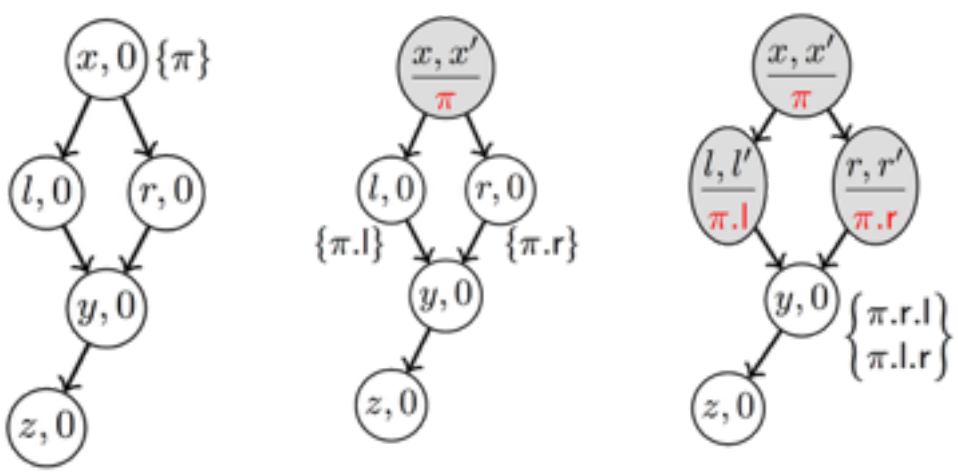

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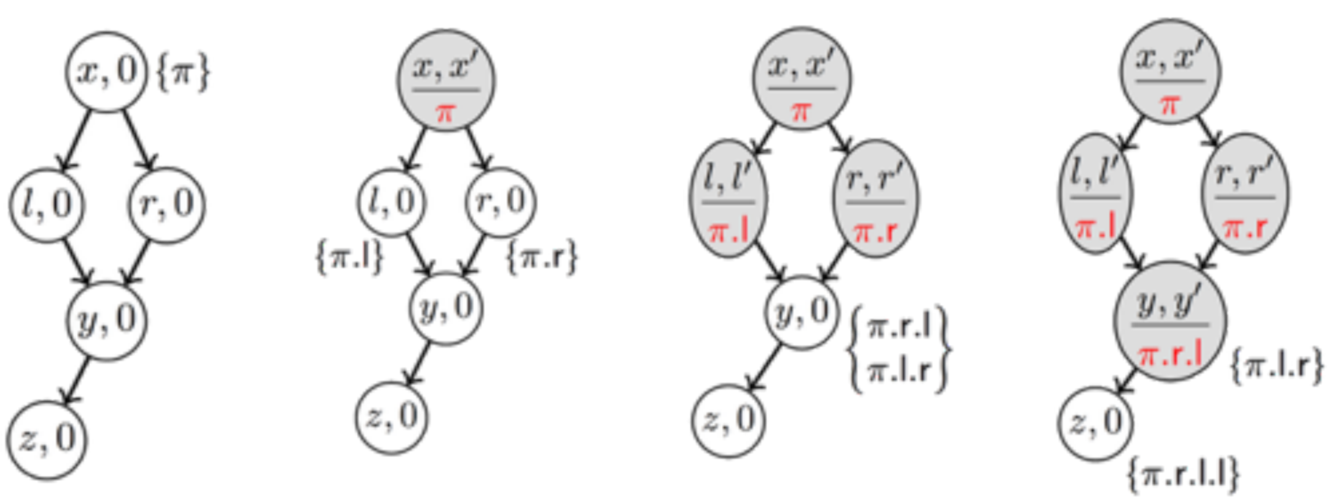

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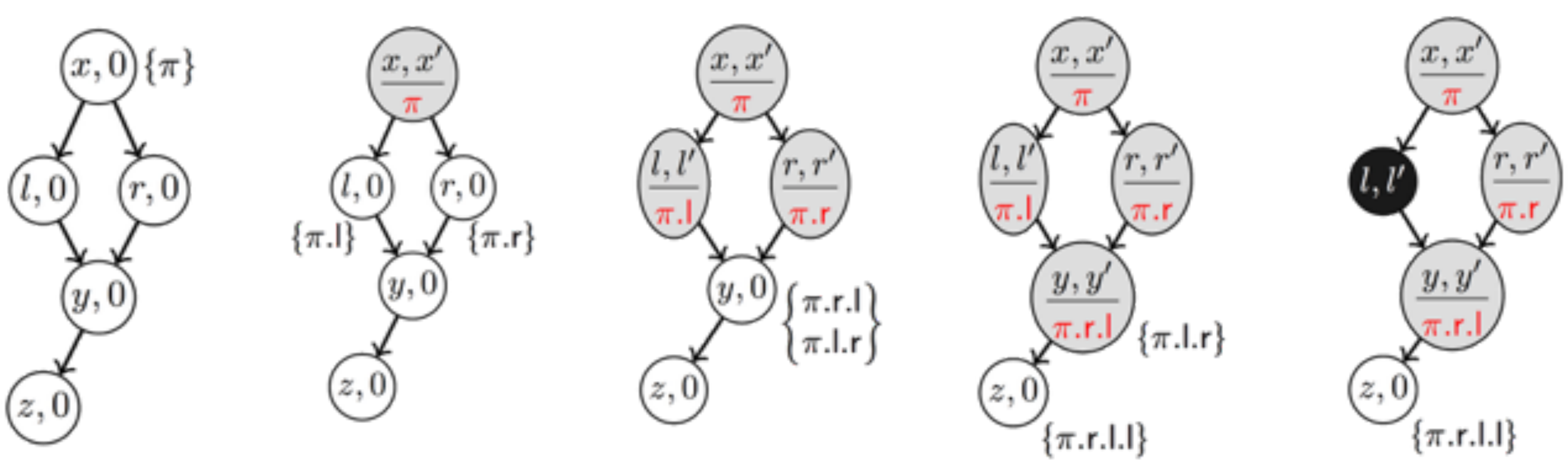

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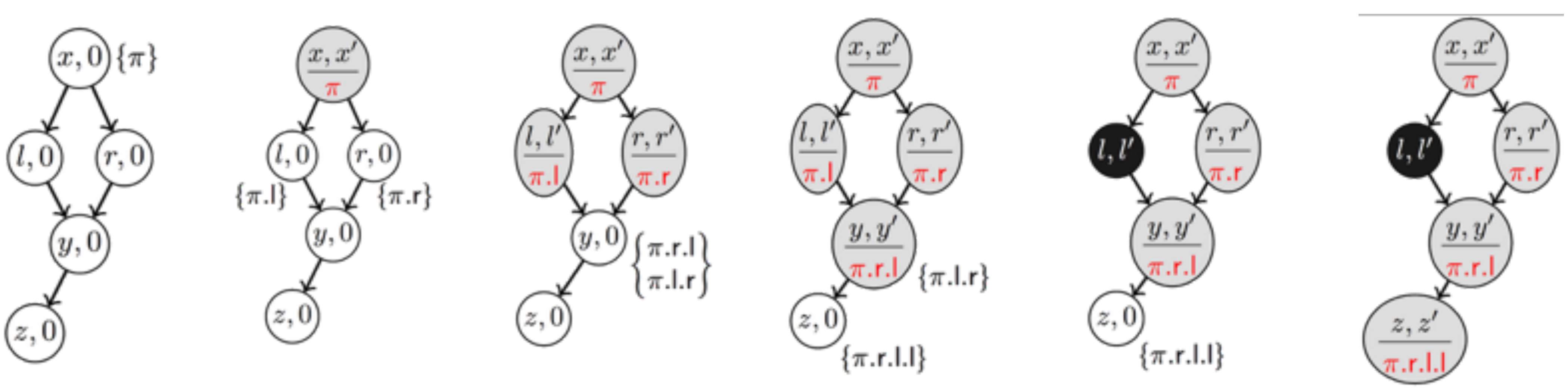

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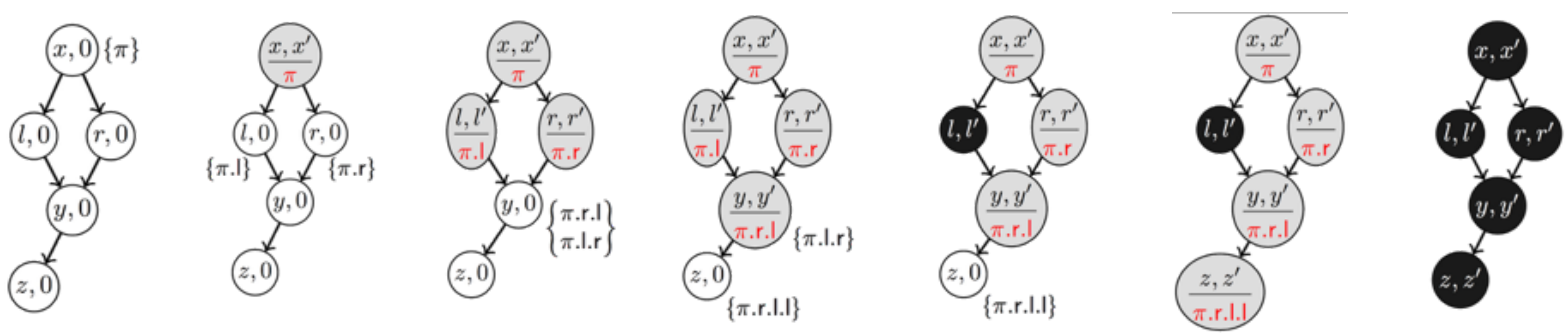

1. Tokens

A token mechanism for

- Thread identification
- Thread progress tracking

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The `copy_dag` token mechanism for

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  - distinguish one token (thread) from another
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  - model a parent-child relation (spawner-spawnee)
- Thread progress tracking
  - marking thread ids as tokens
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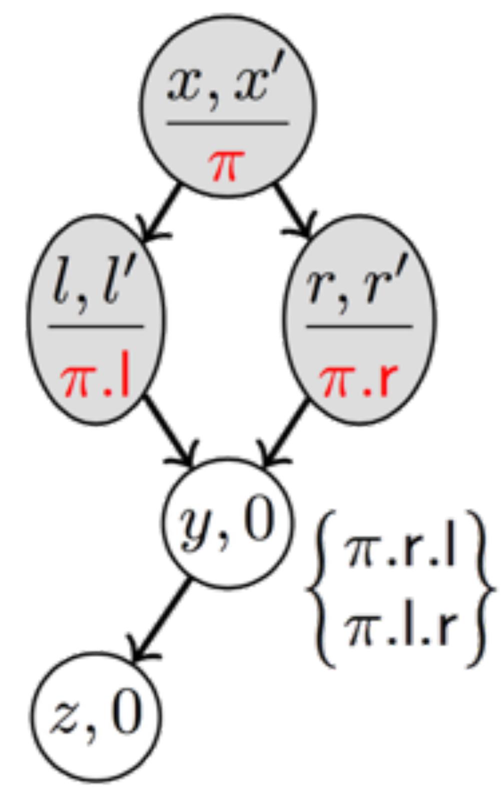

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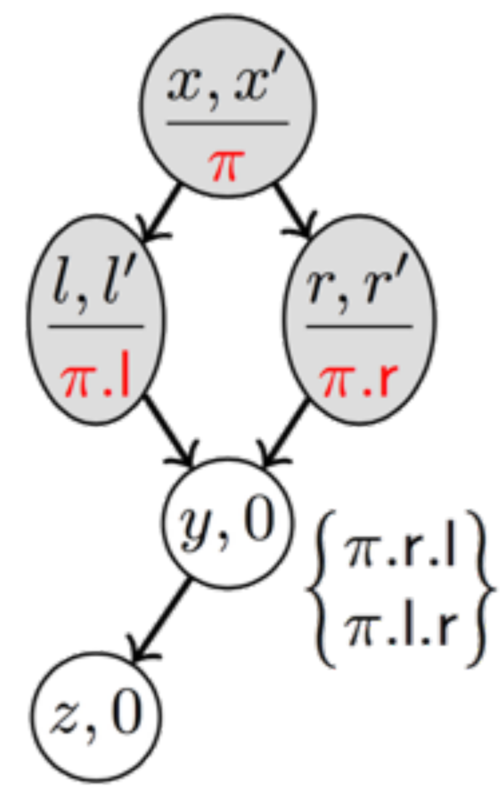

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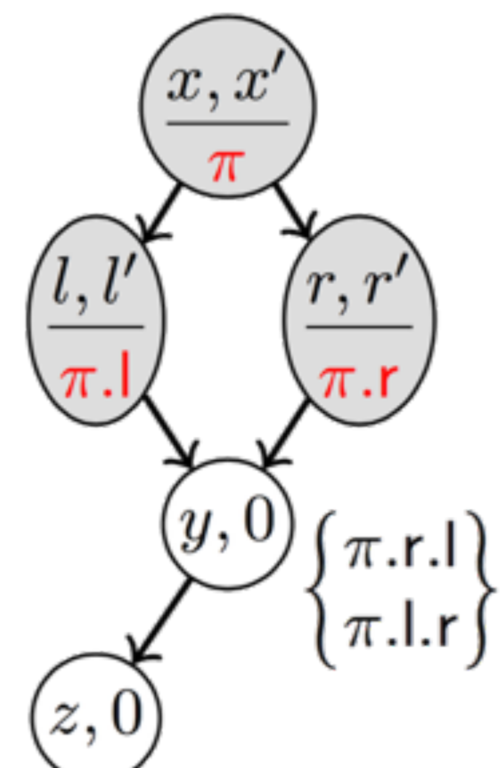

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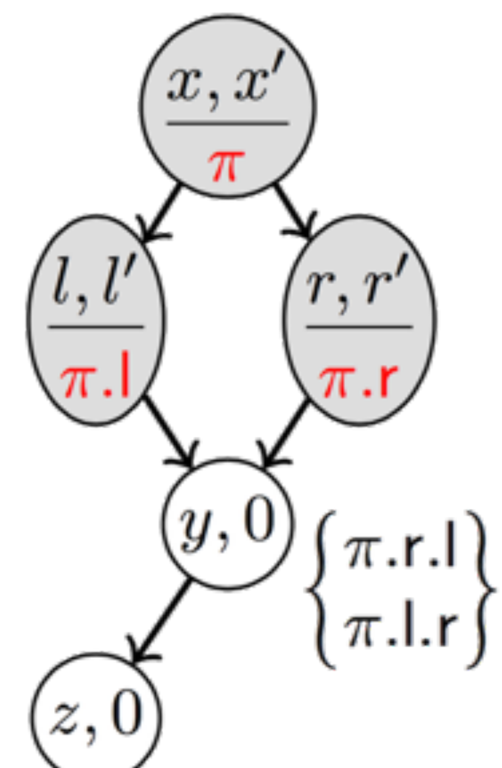

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top-most (maximal) token

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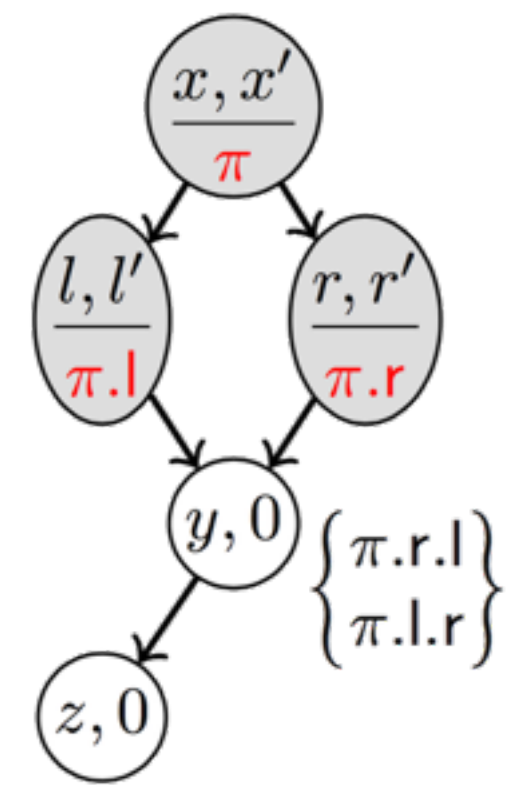

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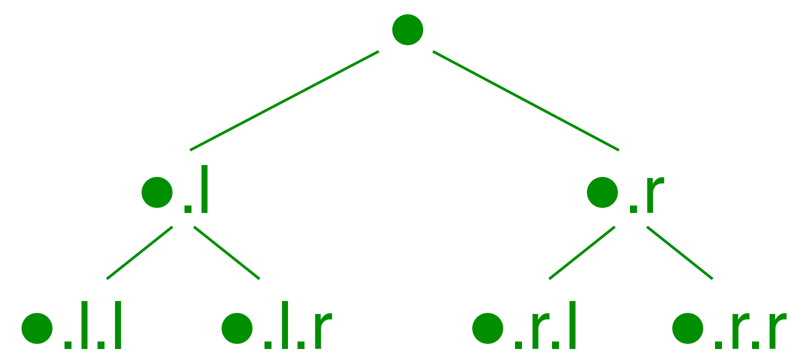

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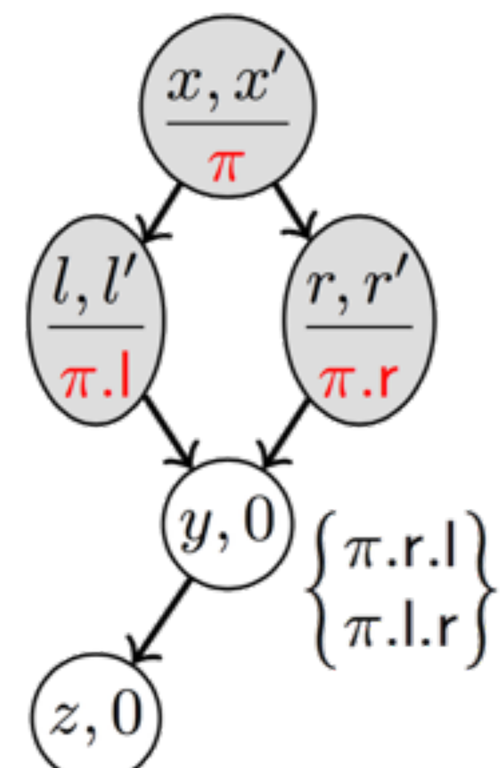

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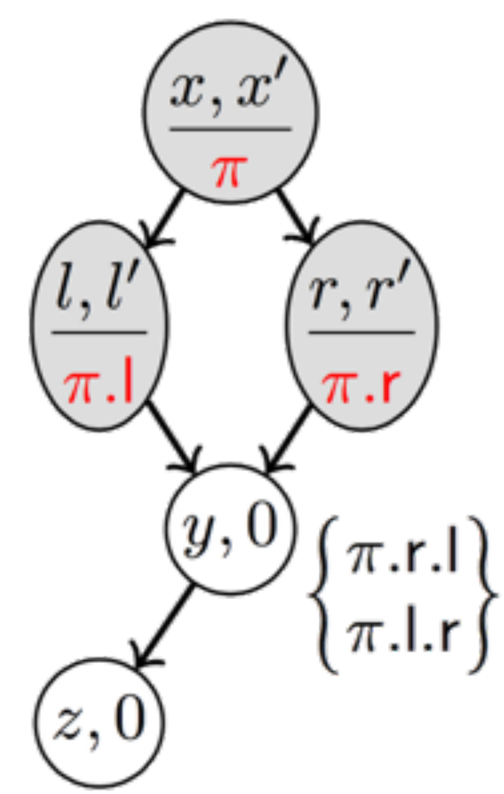
  - each dag is a triple:

$$\delta = (V , E , L)$$



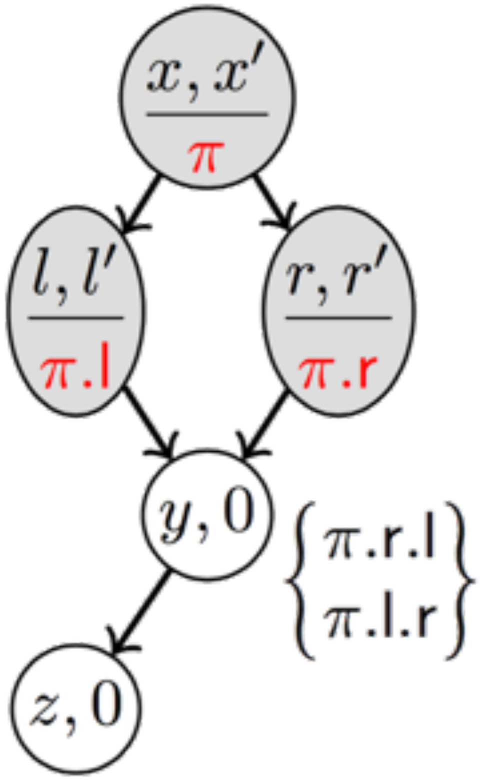
2. Mathematical Objects

- An abstract representation of the underlying data structure
  - e.g. a pair of mathematical dags (δ, δ_c)
  - each dag is a triple:

Vertices

$$\delta = (V , E , L)$$

$$V = \{x,\ l,\ r,\ y,\ z\}$$



2. Mathematical Objects

- An abstract representation of the underlying data structure
  - e.g. a pair of mathematical dags (δ, δ_c)
  - each dag is a triple:

Edges

$$\delta = (V , E , L)$$

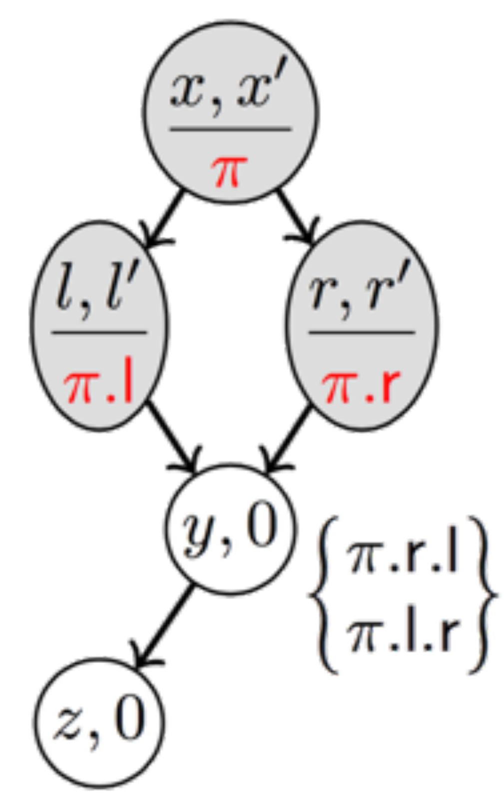
$$E(x) = l,\ r$$

$$E(l) = 0,\ y$$

$$E(r) = y,\ 0$$

$$E(y) = z,\ 0$$

$$E(z) = 0,\ 0$$



2. Mathematical Objects

- An abstract representation of the underlying data structure
  - e.g. a pair of mathematical dags (δ, δ_c)
  - each dag is a triple:

Labels

$$\delta = (V , E , L)$$

copy

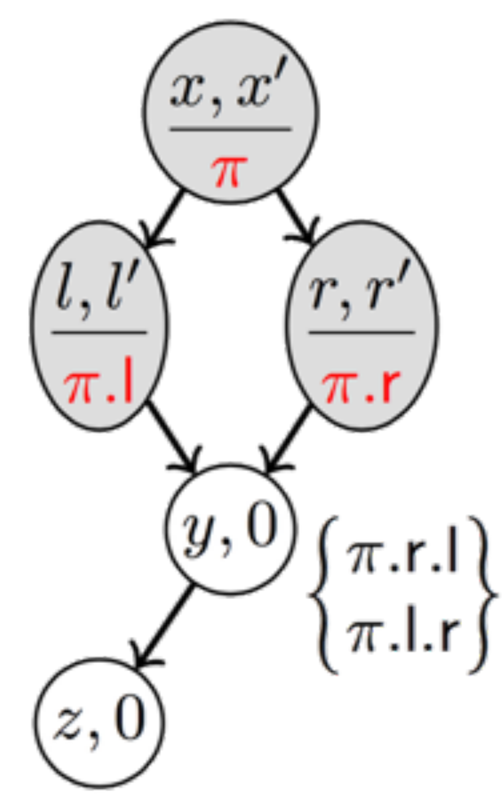
$$L\,(x) = x', \;\pi, \;\{\,\}$$

$$L\,(l) = l', \;\pi.\mathrm{l}, \;\{\,\}$$

$$L\,(r) = r', \;\pi.\mathrm{r}, \;\{\,\}$$

$$L\,(y) = 0, \;0, \;\{\pi.\mathrm{r.l}, \;\pi.\mathrm{l.r}\}$$

$$L\,(z) = 0, \;0, \;\{\,\}$$



2. Mathematical Objects

- An abstract representation of the underlying data structure
  - e.g. a pair of mathematical dags (δ, δ_c)
  - each dag is a triple:

Labels

$$\delta = (V, E, L)$$

copy   copying thread

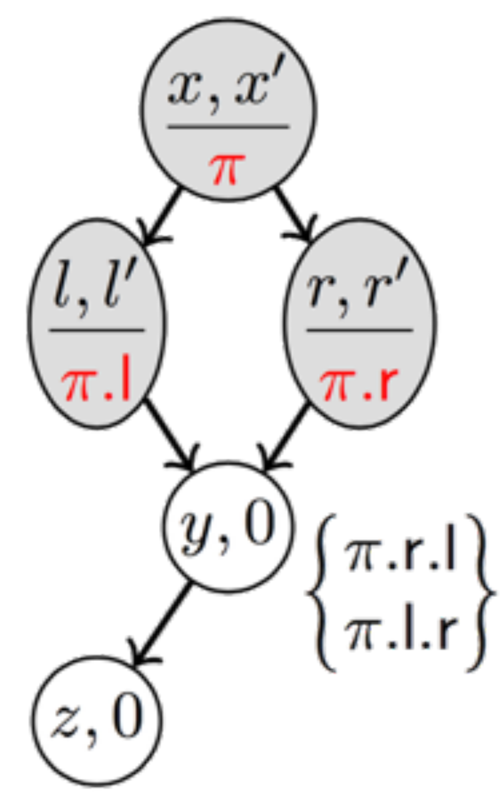
$$L(x) = x',\ \pi,\ \{\ \}$$

$$L(l) = l',\ \pi.\mathrm{l},\ \{\ \}$$

$$L(r) = r',\ \pi.\mathrm{r},\ \{\ \}$$

$$L(y) = 0,\ 0,\ \{\pi.\mathrm{r.l},\ \pi.\mathrm{l.r}\}$$

$$L(z) = 0,\ 0,\ \{\ \}$$



2. Mathematical Objects

- An abstract representation of the underlying data structure
  - e.g. a pair of mathematical dags (δ, δ_c)
  - each dag is a triple:

Labels

$$\delta = (V , E , L)$$

copy copying thread promise set

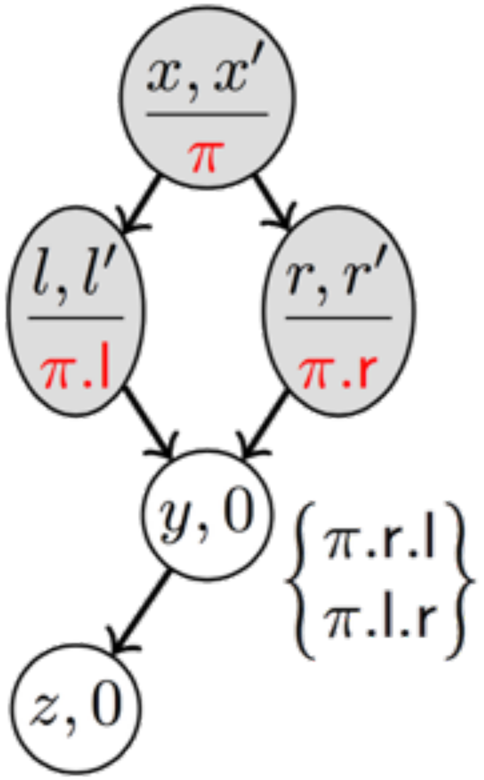
$$L(x) = x', \pi, \{\}$$

$$L(l) = l', \pi.\mathrm{l}, \{\}$$

$$L(r) = r', \pi.\mathrm{r}, \{\}$$

$$L(y) = 0, 0, \{\pi.\mathrm{r.l}, \pi.\mathrm{l.r}\}$$

$$L(z) = 0, 0, \{\}$$



2. Mathematical Objects

- An abstract representation of the underlying data structure
  - e.g. a pair of mathematical dags (δ, δ_c)
  - each dag is a triple:

Labels

$$\delta = (V, E, L)$$

copy copying thread promise set

$$L(x) = x', \pi, \{\}$$

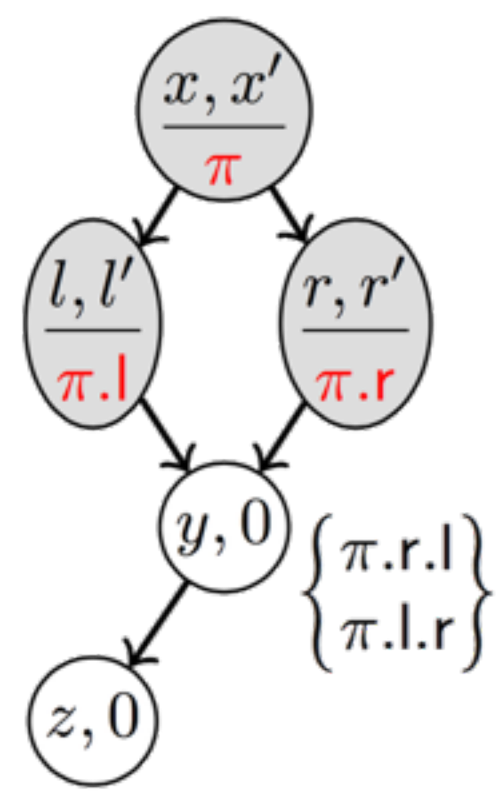
ghost components

$$L(l) = l', \pi.\mathsf{l}, \{\}$$

$$L(r) = r', \pi.\mathsf{r}, \{\}$$

$$L(y) = 0, 0, \{\pi.\mathsf{r}.\mathsf{l}, \pi.\mathsf{l}.\mathsf{r}\}$$

$$L(z) = 0, 0, \{\}$$



2. Mathematical Objects

- An abstract representation of the underlying data structure
  - e.g. a pair of mathematical dags (δ, δ_c)
  - each dag is a triple:

Labels

$$\delta = (V , E , L)$$

copy(x) thread(x) promise(x)

$$L\,(x) = x', \ \pi, \ \{\ \}$$

$$L\,(l) = l', \ \pi.\mathrm{l}, \ \{\ \}$$

$$L\,(r) = r', \ \pi.\mathrm{r}, \ \{\ \}$$

$$L\,(y) = 0, \ 0, \ \{\pi.\mathrm{r.l}, \ \pi.\mathrm{l.r}\}$$

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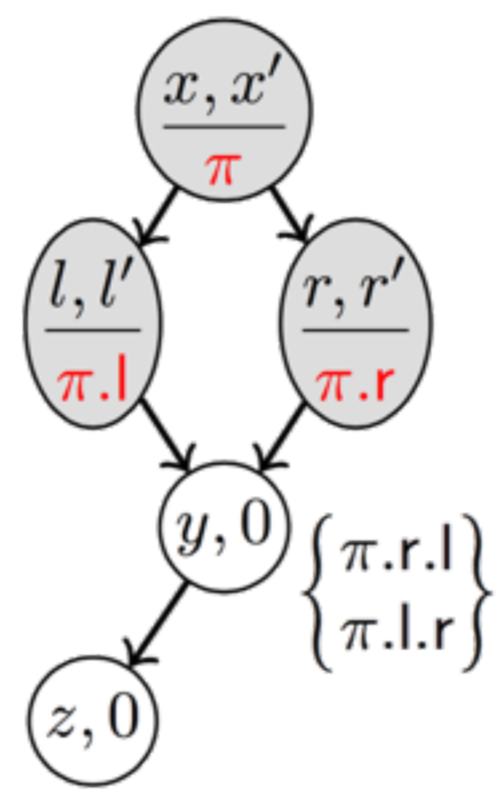


3. Mathematical Actions

- An abstraction of thread actions (on mathematical objects)
  - atomic blocks as well as ghost actions
  - A^π denotes the actions of thread π

3. Mathematical Actions

- An abstraction of thread actions (on mathematical objects)
  - atomic blocks as well as ghost actions
  - A^{π} denotes the actions of thread π

```
struct node {struct node *c, *l, *r}
copy_dag(struct node *x) {
  struct node *l, *r, *ll, *rr, *x';  bool b;
  if (!x) {return 0;}
  x' = malloc(sizeof(struct node));
  b = <CAS(x->c, 0, x')>;
  if (b) {
    l = x->l; r = x->r;
    ll = copy_dag(l)  ||  rr = copy_dag(r)
    <x'->l = ll>; <x'->r = rr>;
    return x';
  } else {
    free(x', sizeof(struct node));  return x->c;
  }
}
```

3. Mathematical Actions

- An abstraction of thread actions (on mathematical objects)
  - atomic blocks as well as ghost actions
  - A^{π} denotes the actions of thread π

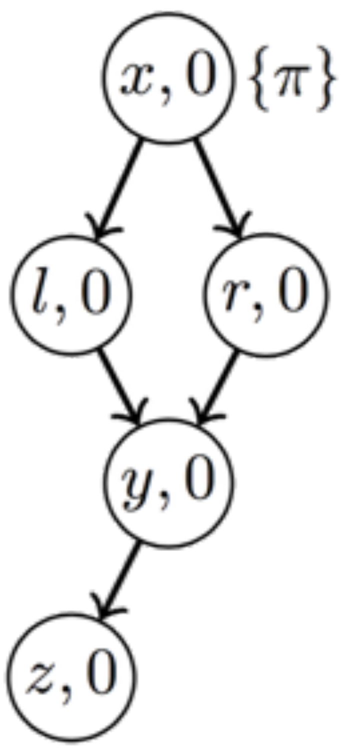
```
struct
copy_dag(
  struct
  if
  x' =
  b = <CAS(x->c, 0, x')>;
  if
    l = x->l; r = x->r;
    ll = copy_dag(l)  ||  rr = copy_dag(r)
    <x'->l = ll>; <x'->r = rr>

  } else {

  }
}
```

3. Mathematical Actions

- An abstraction of thread actions (on mathematical objects)
  - atomic blocks as well as ghost actions
  - A^{π} denotes the actions of thread π




```
b = <CAS(x->c, 0, x')>;
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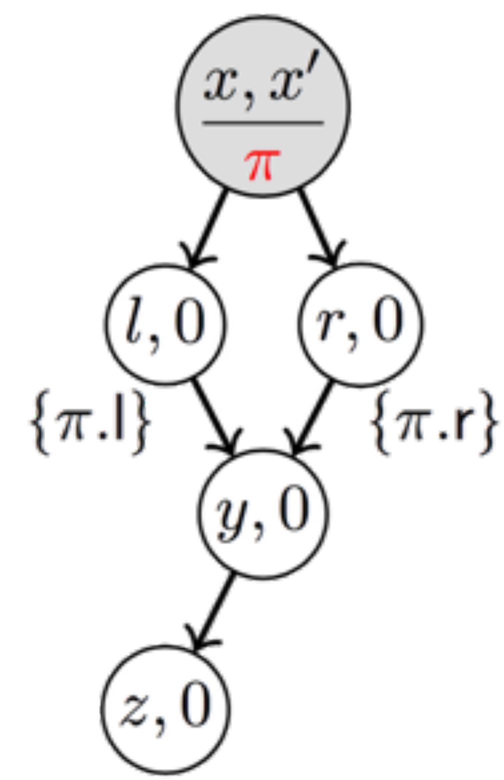


$(\delta, \delta_C) = ((V, E, L), (V_C, E_C, L_C))$ $\overset{A^{\pi}}{\rightsquigarrow}$ $(\delta', \delta'_C) = ((V, E, L'), (V'_C, E'_C, L'_C))$

$L(x) = 0,\ 0,\ \{\pi\}$
$L(l) = 0,\ 0,\ \{\ \}$
$L(r) = 0,\ 0,\ \{\ \}$

$L' = L[x \mapsto x',\ \pi,\ \{\ \}][l \mapsto 0,0,\{\pi.l\}][r \mapsto 0,0,\{\pi.r\}]$

$V'_C = V_C \uplus \{x'\}$ $\quad E'_C = E_C \uplus [x' \mapsto \cdots]$ $\quad L'_C = L_C \uplus [x' \mapsto \cdots]$

4. Mathematical Specification

Inv(δ, $δ_c$) ≜ δ and $δ_c$ are both acyclic;
every node x' in the copy $δ_c$ corresponds to a unique node x in the original δ;
every node x in the original δ has some copy value x'

4. Mathematical Specification

$\mathrm{Inv}(\delta, \delta_c) \triangleq \mathrm{acyc}(\delta) \wedge \mathrm{acyc}(\delta_c) \wedge$
every node x' in the copy δ_c corresponds to a unique node x in the original δ;
every node x in the original δ has some copy value x'

4. Mathematical Specification

$\mathrm{Inv}(\delta, \delta_c) \triangleq \mathrm{acyc}(\delta) \wedge \mathrm{acyc}(\delta_c) \wedge \forall x' \in \delta_c.\ \exists! x \in \delta.\ \mathrm{copy}(x) = x' \wedge$

every node x in the original δ has some copy value x'

4. Mathematical Specification

$$\mathrm{Inv}(\delta, \delta_c) \triangleq \mathrm{acyc}(\delta) \land \mathrm{acyc}(\delta_c) \land \forall x' \in \delta_c.\ \exists! x \in \delta.\ \mathrm{copy}(x) = x'$$
$$\land\ \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \land \mathrm{ic}(x, x', \delta, \delta_c)$$

4. Mathematical Specification

$\mathrm{Inv}(\delta, \delta_c) \triangleq \mathrm{acyc}(\delta) \land \mathrm{acyc}(\delta_c) \land \forall x' \in \delta_c.\ \exists! x \in \delta.\ \mathrm{copy}(x) = x'$
$\land\ \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \land \mathrm{ic}(x, x', \delta, \delta_c)$

$\mathrm{ic}(x, x', \delta, \delta_c) \triangleq$ if x' is 0 (x is not copied yet), then x will eventually be copied:

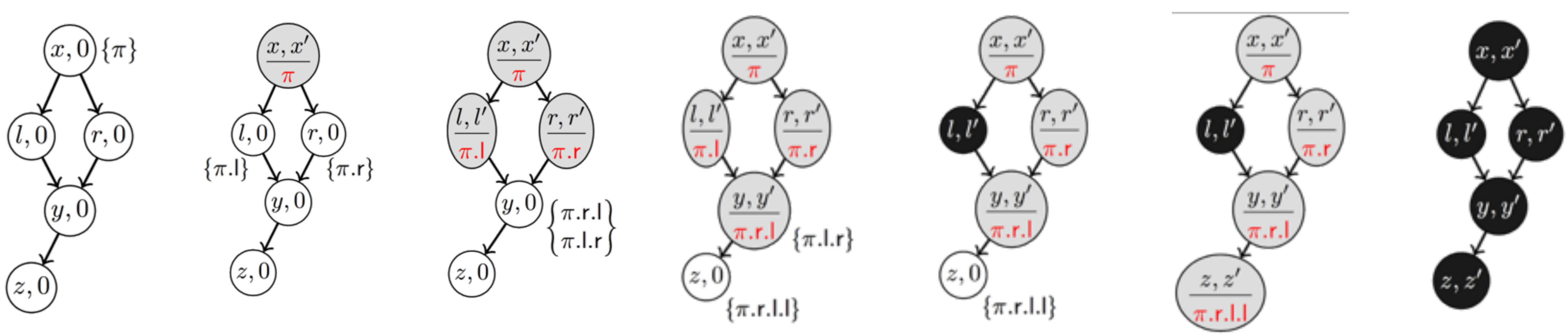
4. Mathematical Specification

Inv(δ, δ_c) ≜ acyc(δ) ∧ acyc(δ_c) ∧ $\forall x' \in \delta_c.\ \exists! x \in \delta.$ copy(x)= x'
∧ $\forall x \in \delta.\ \exists x'.$ copy(x)=x' ∧ ic(x, x', δ, δ_c)

ic(x, x', δ, δ_c) ≜ if x' is 0 (x is not copied yet), then x will eventually be copied:

there exists some y in δ s.t.
1) the promise set of y is non-empty; 2) y can reach x along a path p;
and 3) every node along the path p is not copied

⇒ when y is eventually copied, it'll visit x along p and copy it too



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Inv(δ, δ_c) ≜ acyc(δ) ∧ acyc(δ_c) ∧ $\forall x' \in \delta_c.\ \exists! x \in \delta.$ copy(x)= x'

∧ $\forall x \in \delta.\ \exists x'.$ copy(x)=x' ∧ ic(x, x', δ, δ_c)

ic(x, x', δ, δ_c) ≜ if x' is 0 (x is not copied yet), then x will eventually be copied:

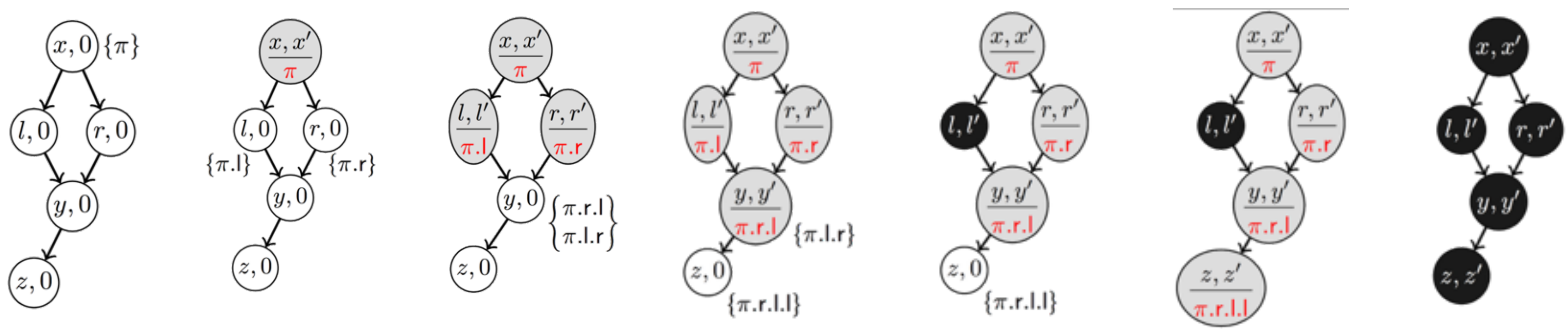
there exists some y in δ s.t.

1) the promise set of y is non-empty; 2) y can reach x along a path p; and 3) every node along the path p is not copied

otherwise, x' is a node in δ_c and

the children of x, (l,r), are also copied to some (l',r'):

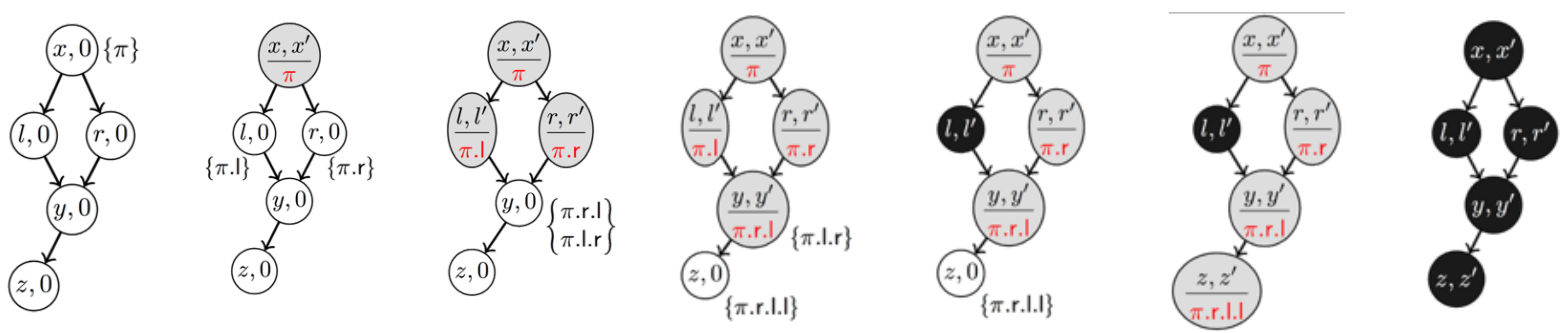
ic(l, l', δ, δ_c) and ic(r, r', δ, δ_c)



4. Mathematical Specification

$\mathrm{Inv}(\delta, \delta_c) \triangleq \mathrm{acyc}(\delta) \wedge \mathrm{acyc}(\delta_c) \wedge \forall x' \in \delta_c.\ \exists! x \in \delta.\ \mathrm{copy}(x) = x'$
$\wedge\ \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \wedge \mathrm{ic}(x, x', \delta, \delta_c)$

$$
\begin{aligned}
\mathrm{ic}(x, x', \delta, \delta_c) \triangleq\ & (x=0 \wedge x'=0) \vee \\
& \Big(x \neq 0 \wedge \big[(x'=0 \wedge \delta^{\mathsf{c}}(x)=x' \wedge \exists y.\ \delta^{\mathsf{p}}(y) \neq \varnothing \wedge y \overset{\delta}{\rightsquigarrow}{}^{*}_{0}\ x) \\
& \qquad \vee (x' \neq 0 \wedge x' \in \delta' \wedge \exists \pi, l, r, l', r'.\ \delta(x) = ((x', \pi, -), l, r) \wedge \delta'(x') = (-, l', r') \\
& \qquad\quad \wedge (l' \neq 0 \Rightarrow \mathrm{ic}(l, l', \delta, \delta')) \wedge (r' \neq 0 \Rightarrow \mathrm{ic}(r, r', \delta, \delta'))) \\
& \qquad \vee (x' \neq 0 \wedge x' \in \delta' \wedge \exists l, r, l', r'.\ \delta(x) = ((x', 0, -), l, r) \wedge \delta'(x') = (-, l', r') \\
& \qquad\quad \wedge \mathrm{ic}(l, l', \delta, \delta') \wedge \mathrm{ic}(r, r', \delta, \delta'))\big]\Big)
\end{aligned}
$$



4. Mathematical Specification

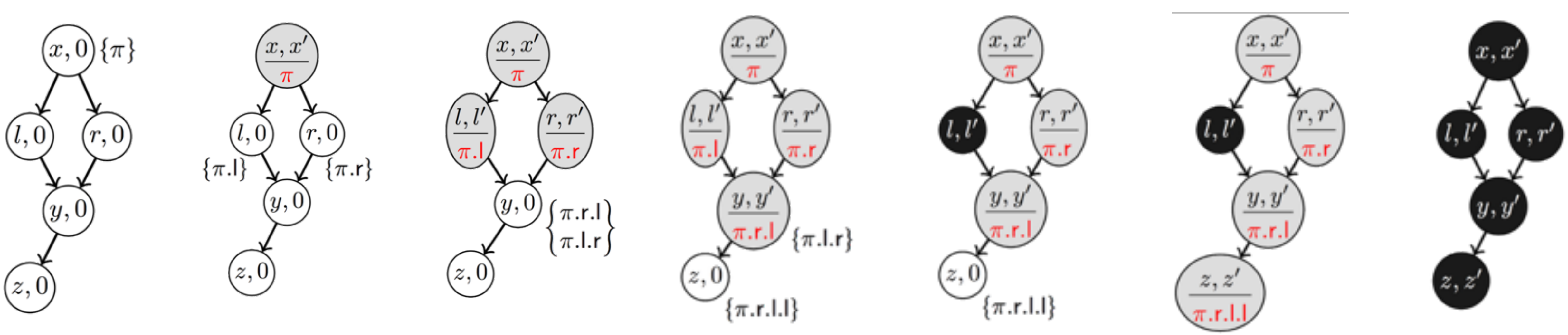
Inv(δ, δ_c) ≜ acyc(δ) ∧ acyc(δ_c) ∧ $\forall x' \in \delta_c.\ \exists! x \in \delta.$ copy(x)= x'

∧ $\forall x \in \delta.\ \exists x'.$ copy(x)=x' ∧ ic(x, x', δ, δ_c)

$P^{\pi}(x, \delta)$ ≜ π has made a promise to visit x; π has made a promise to x only; and

π has not spawned any threads yet:

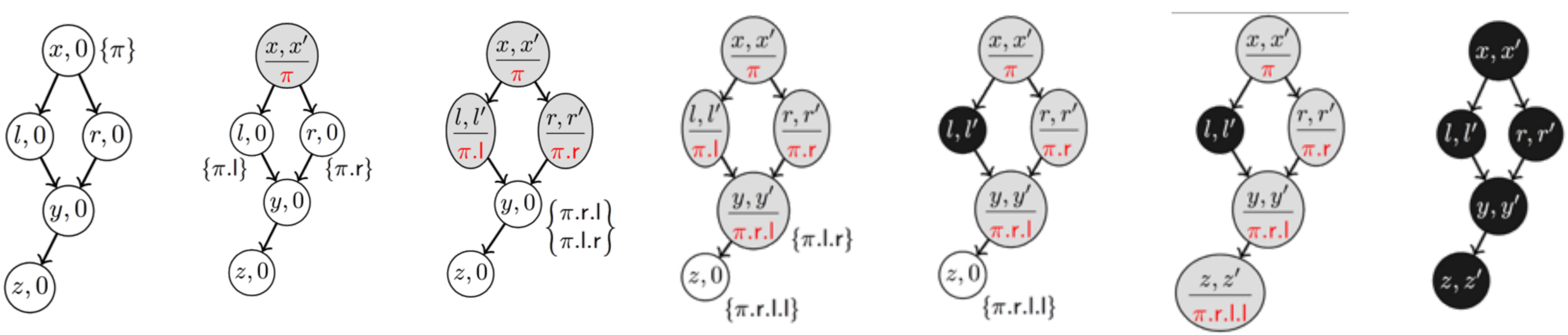
its subthreads are not in the graph (in promise sets or as copying thread)



4. Mathematical Specification

Inv(δ, δ_c) ≜ acyc(δ) ∧ acyc(δ_c) ∧ $\forall x' \in \delta_c.\ \exists! x \in \delta.$ copy(x)= x'
∧ $\forall x \in \delta.\ \exists x'.$ copy(x)=x' ∧ ic(x, x', δ, δ_c)

$P^{\pi}(x, \delta) \triangleq x \neq 0 \Rightarrow \pi \in$ promise(x) ∧ π has made a promise to x only; and
π has not spawned any threads yet:
its subthreads are not in the graph (in promise sets or as copying thread)



4. Mathematical Specification

$\mathsf{Inv}(\delta, \delta_c) \triangleq \mathsf{acyc}(\delta) \land \mathsf{acyc}(\delta_c) \land \forall x' \in \delta_c.\ \exists! x \in \delta.\ \mathsf{copy}(x) = x'$

$\land\ \forall x \in \delta.\ \exists x'.\ \mathsf{copy}(x) = x' \land \mathsf{ic}(x, x', \delta, \delta_c)$

$\mathsf{P}^{\pi}(x, \delta) \triangleq x \neq 0 \Rightarrow \pi \in \mathsf{promise}(x) \quad \land \quad \forall z \in \delta.\ \pi \in \mathsf{promise}(z) \Rightarrow x = z$

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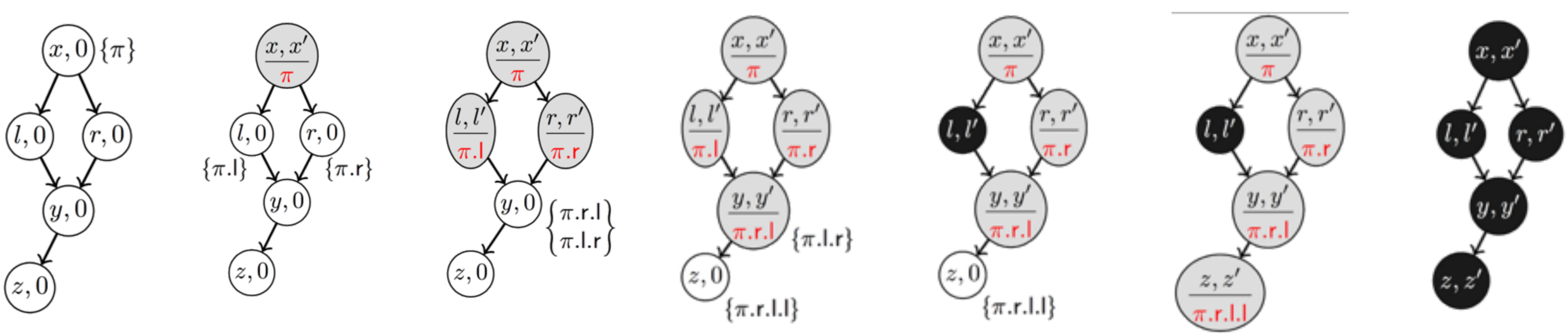

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$$\mathrm{Inv}(\delta, \delta_c) \triangleq \mathrm{acyc}(\delta) \wedge \mathrm{acyc}(\delta_c) \wedge \forall x' \in \delta_c.\ \exists! x \in \delta.\ \mathrm{copy}(x) = x'$$
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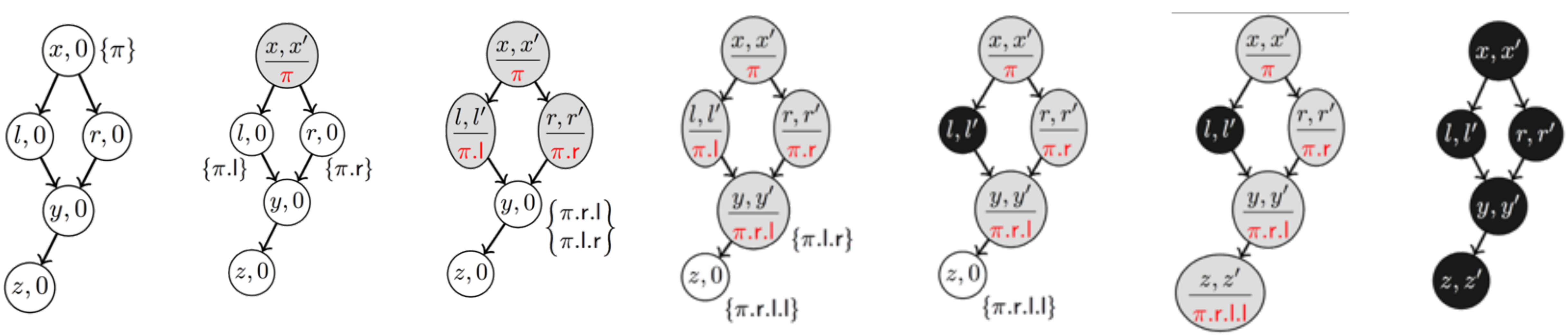
$$P^{\pi}(x, \delta) \triangleq x \neq 0 \Rightarrow \pi \in \mathrm{promise}(x) \wedge \forall z \in \delta.\ \pi \in \mathrm{promise}(z) \Rightarrow x = z$$
$$\forall z \in \delta.\ \forall \pi' \sqsubset \pi.\ \pi' \notin \mathrm{promise}(z) \wedge \pi' \neq \mathrm{thread}(x)$$



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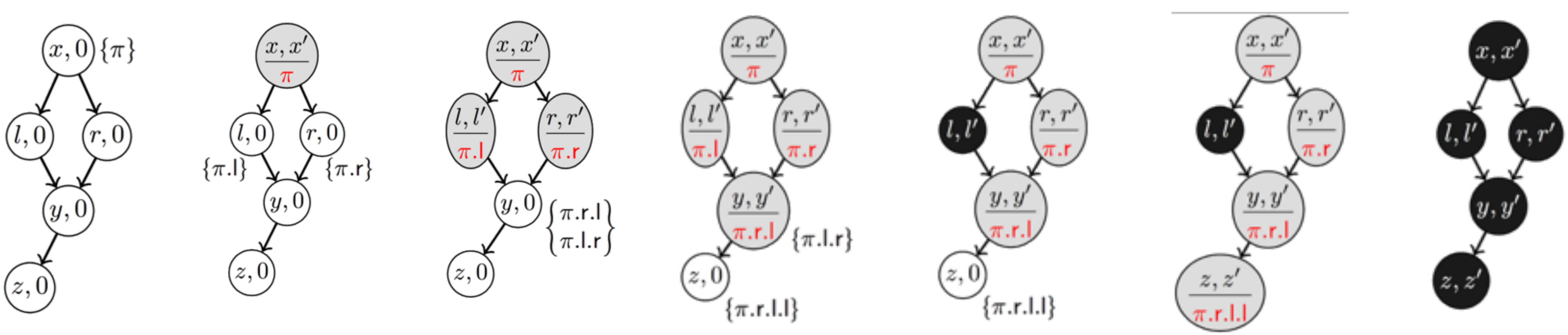
$\wedge\ \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \wedge \mathrm{ic}(x, x', \delta, \delta_c)$

$P^{\pi}(x, \delta) \triangleq x \neq 0 \Rightarrow \pi \in \mathrm{promise}(x) \wedge \forall z \in \delta.\ \pi \in \mathrm{promise}(z) \Rightarrow x = z$
$\forall z \in \delta.\ \forall \pi' \sqsubset \pi.\ \pi' \notin \mathrm{promise}(z) \wedge \pi' \neq \mathrm{thread}(x)$

$Q^{\pi}(x, x', \delta, \delta_c) \triangleq$ x is copied to x' in δ_c ; and
π and all its subthreads have finished executing (have joined):
they are not in the graph (in promise sets or as copying thread)



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$$\mathsf{P}^{\pi}(x, \delta) \triangleq x \neq 0 \Rightarrow \pi \in \mathsf{promise}(x) \wedge \forall z \in \delta.\ \pi \in \mathsf{promise}(z) \Rightarrow x = z$$
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$$\mathsf{Q}^{\pi}(x, x', \delta, \delta_c) \triangleq \mathsf{copy}(x) = x' \wedge x' \in \delta_c \wedge$$

π and all its subthreads have finished executing (have joined):

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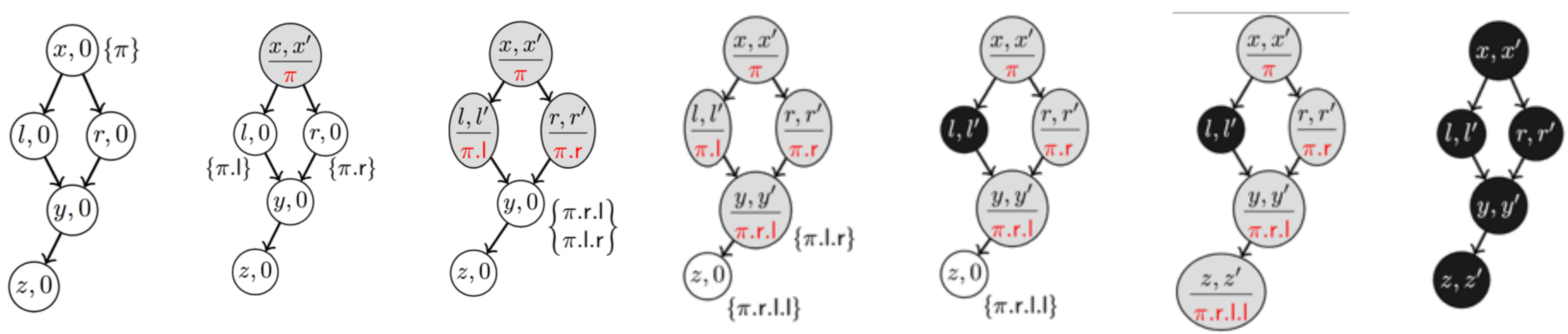

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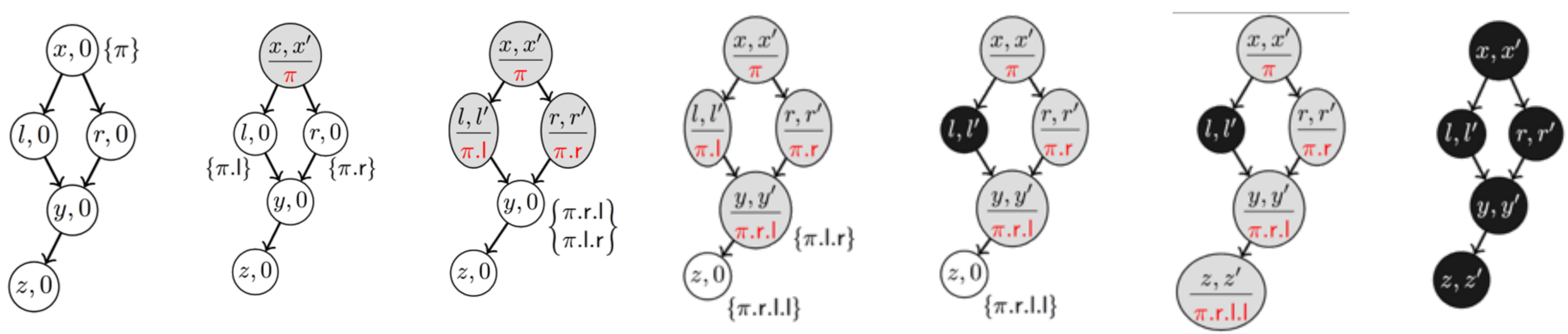

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$$Q^{\bullet}(x, x', \delta, \delta_c) \triangleq \mathrm{copy}(x) = x' \land x' \in \delta_c \land$$
$$\forall z \in \delta.\ \forall \pi' \sqsubseteq \bullet.\quad \pi' \notin \mathrm{promise}(z) \land \pi' \neq \mathrm{thread}(x)$$

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$Q^{\bullet}(x, x', \delta, \delta_c) \triangleq \mathrm{copy}(x) = x' \wedge x' \in \delta_c \wedge$
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$\forall\ \pi.\ \pi \sqsubseteq \bullet$

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$\mathrm{ic}(x, x', \delta, \delta_c) \triangleq$ if x' is 0 (x is not copied yet), then x will eventually be copied:

there exists some y in δ s.t.

1) the promise set of y is non-empty; 2) y can reach x along a path p;
and 3) every node along the path p is not copied

otherwise, x' is a node in δ_c and the children of x are also copied

4. Mathematical Specification

$\mathrm{Inv}(\delta, \delta_c) \triangleq \mathrm{acyc}(\delta) \wedge \mathrm{acyc}(\delta_c) \wedge \forall x' \in \delta_c.\ \exists! x \in \delta.\ \mathrm{copy}(x) = x'$
$\wedge\ \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \wedge \mathrm{ic}(x, x', \delta, \delta_c)$

$Q^{\bullet}(x, x', \delta, \delta_c) \triangleq \mathrm{copy}(x) = x' \wedge x' \in \delta_c \wedge$
$\forall z \in \delta.\ \forall \pi'.\quad \pi' \notin \mathrm{promise}(z) \wedge \pi' \neq \mathrm{thread}(x)$

$\mathrm{ic}(x, x', \delta, \delta_c) \triangleq$ if x' is 0 (x is not copied yet), then x will eventually be copied:
there exists some y in δ s.t.
1) the promise set of y is non-empty; 2) y can reach x along a path p;
and 3) every node along the path p is not copied
otherwise, x' is a node in δ_c and the children of x are also copied

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$\mathrm{ic}(x, x', \delta, \delta_c) \triangleq$ x' is a node in δ_c and the children of x are also copied

$Q^{\bullet}(x, x', \delta, \delta_c) \wedge \mathrm{ic}(x, x', \delta, \delta_c) \Rightarrow$ all nodes in δ are copied to nodes in δ_c

5. Spatial Objects

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- A concrete implementation of the data structures in the heap

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recorded in the *ghost heap; not "baked in"* to model

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Tracking the abstract state of the dags:
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$$\mathrm{node}(x, (V, E, L)) \triangleq \exists l, r, x', P, \pi.\ E(x) = l, r \wedge L(x) = x', \pi, P \wedge x \mapsto x', l, r \ast x \Rrightarrow \pi, P$$

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- An implementation of thread actions (on spatial objects)
  - atomic blocks as well as ghost actions

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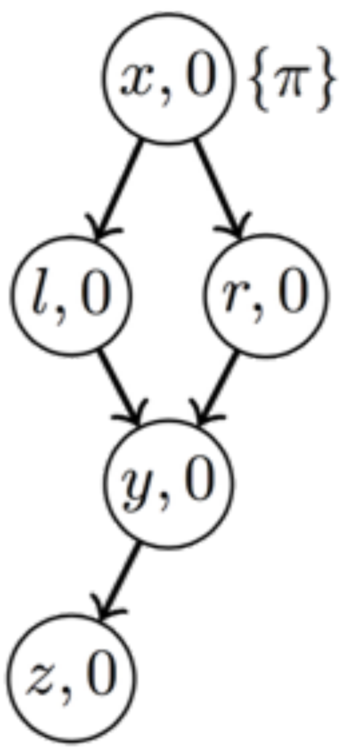
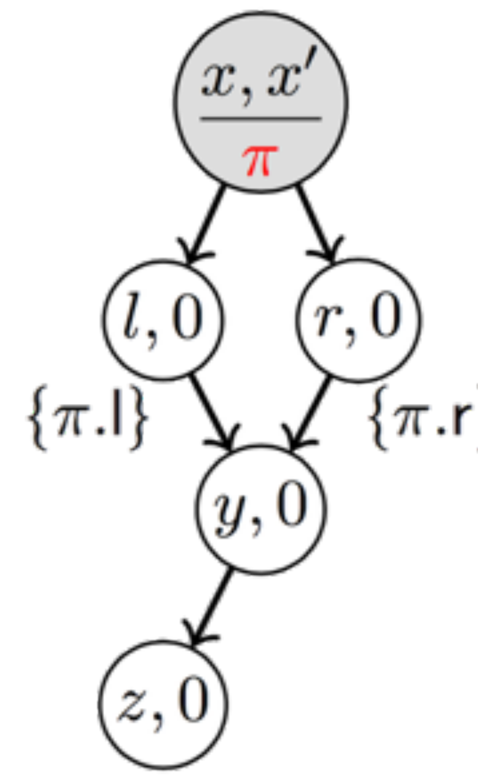



```
struct
copy_dag(
  struct
  if
  x' =
  b = <CAS(x->c, 0, x')>;
  if
    l = x->l; r = x->r;
    ll = copy_dag(l)  ||  rr = copy_dag(r)
    <x'->l = ll>; <x'->r = rr>

  } else {

  }
}
```



(δ, δ_c) $\quad \overset{A^{\pi}}{\rightsquigarrow} \quad$ (δ', δ'_c)

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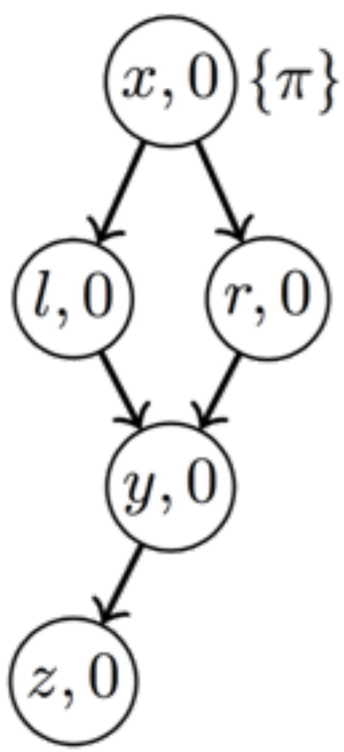


```
struct
copy_dag(
  struct
  if
  x' =
  b = <CAS(x->c, 0, x')>;
  if
    l = x->l; r = x->r;
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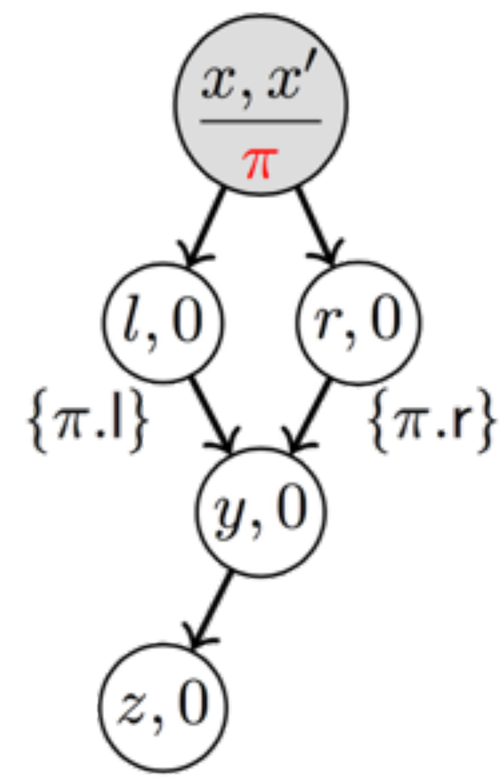
    <x'->l = ll>; <x'->r = rr>

  } else {

  }
}
```



(δ, δ_c) $\qquad \overset{A^{\pi}}{\rightsquigarrow} \qquad$ (δ’, δ’$_c$)

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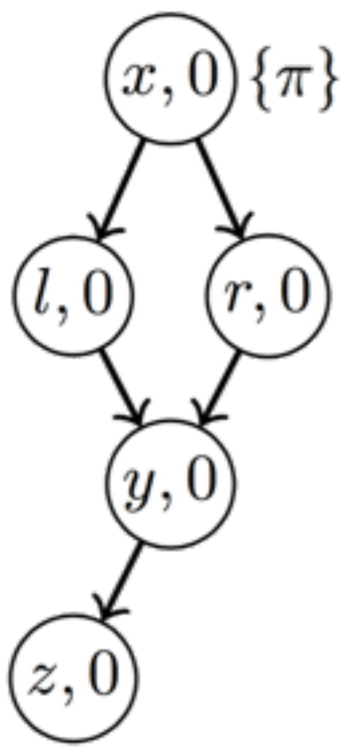


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  if
  x' =
b = <CAS(x->c, 0, x')>;
  if
    l = x->l; r = x->r;
    ll = copy_dag(l)  ||  rr = copy_dag(r)
    <x'->l = ll>; <x'->r = rr>

  } else {

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```

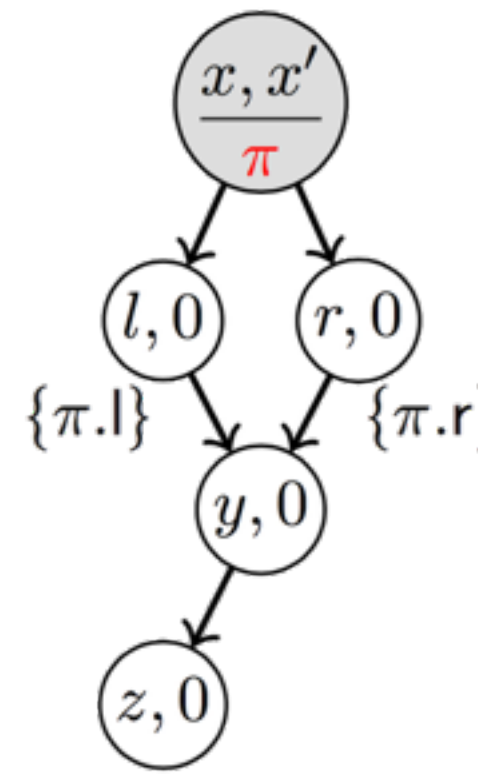

(δ, δc) $\overset{A^{\pi}}{\rightsquigarrow}$ (δ', δ'c)

icdag(δ, δc) $\overset{[A^{\pi}]}{\rightsquigarrow}$ icdag(δ', δ'c)

Verifying `copy_dag(x)`

```
struct node {struct node *c, *l, *r};
{Pre(x, π, δ)}
copy_dag(struct node *x) {struct node *l, *r, *ll, *rr, *y; bool b;
{π̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^π(x, δ1))]_I }
  if(!x){ return 0; }
{π̄• • ret≐0 • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, ret, δ1, δ2))]_I }
  y = malloc(sizeof(struct node));
{π̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^π(x, δ1))]_I • y ↦ 0,0,0 • y ⇛ π, ∅ }
  <if(x->c){ b = false;        //Perform the action A^5_π
{π̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, δ1^c(x), δ1, δ2) ∧ δ1^c(x) ≠ 0)]_I
 • y ↦ 0, −, − • y ⇛ π, ∅ • b≐0 }
  }else{ x->c = y; b = true;     //Perform the action A^1_π
{π̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
   (x≠y ⇒ π≠δ1^s(y)) ∧ ∃l, r. δ1(x)=(y, π, −, l, r) ∧ y∈̇δ2 ∧ P^{π.l}(l, δ1) ∧ P^{π.r}(r, δ1))]_I • b≐1 }
  }>
  if(b){ l = x->l; r = x->r;
{π̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
   (x≠y ⇒ π≠δ1^s(y)) ∧ δ1(x)=(y, π, −, l, r) ∧ y∈̇δ2 ∧ P^{π.l}(l, δ1) ∧ P^{π.r}(r, δ1))]_I }
{π • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
   (x≠y ⇒ π≠δ1^s(y)) ∧ δ1(x)=(y, −, π, l, r) ∧ y∈̇δ2)]_I
 • π.l̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^{π.l}(l, δ1))]_I
 • π.r̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^{π.r}(r, δ1))]_I }
{π • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y)
   ∧ (x≠y ⇒ π≠δ1^s(y)) ∧ δ1(x)=(y, −, π, l, r) ∧ y∈̇δ2)]_I • Pre(l, π.l, δ) • Pre(r, π.r, δ) }
          {Pre(l, π.l, δ)}          ||     {Pre(r, π.r, δ)}
          ll = copy_dag(l)          ||     rr = copy_dag(r)
          {Post(l, ll, π.l, δ)}     ||     {Post(r, rr, π.r, δ)}
{π • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y)
   ∧ (x≠y ⇒ π≠δ1^s(y)) ∧ δ1(x)=(y, −, π, l, r) ∧ y∈̇δ2)]_I • Post(l, ll, π.l, δ) • Post(r, rr, π.r, δ) }
{π̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
   (x≠y ⇒ π≠δ1^s(y)) ∧ δ1(x)=(y, −, π, l, r) ∧ y∈̇δ2 ∧ Q^{π.l}(l, ll, δ1, δ2) ∧ Q^{π.r}(r, rr, δ1, δ2))]_I }
    <y->l = ll>; <y->r = rr>;  //Perform A^2_π, A^3_π and A^4_π in order.
{π̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, y, δ1, δ2))]_I }
    return y;  {π̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, ret, δ1, δ2))]_I }
  }else{
{π̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, δ1^c(x), δ1, δ2) ∧ δ1^c(x) ≠ 0)]_I • y ↦ 0, −, − • y ⇛ π, ∅ }
    free(y, sizeof(struct node)) ; return x->c;
{π̄• • [∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, ret, δ1, δ2))]_I }
} } {Post(x, ret, π, δ)}
```

Changes reflected in the pure (mathematical) part as highlighted

Verifying `copy_dag(x)`

```
struct node {struct node *c, *l, *r};
```

$$\{\mathsf{Pre}(x,\pi,\delta)\}$$

```
copy_dag(struct node *x) {struct node *l,*r,*ll,*rr,*y; bool b;
```

$$\left\{\pi^\bullet * \boxed{\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \mathsf{P}^\pi(x,\delta_1))}_I\right\}$$

```
  if(!x){ return 0; }
```

$$\left\{\pi^\bullet * \mathsf{ret} \doteq 0 * \boxed{\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \mathsf{Q}^\pi(x,\mathsf{ret},\delta_1,\delta_2))}_I\right\}$$

```
  y = malloc(sizeof(struct node));
```

$$\left\{\pi^\bullet * \boxed{\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \mathsf{P}^\pi(x,\delta_1))}_I * \mathsf{y} \mapsto 0,0,0 * \mathsf{y} \Rrightarrow \pi,\emptyset\right\}$$

```
  <if(x->c){ b = false;
```
//Perform the action A^5_π

$$\left\{\begin{array}{l}\pi^\bullet * \boxed{\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \mathsf{Q}^\pi(x,\delta_1^c(\mathsf{x}),\delta_1,\delta_2) \wedge \delta_1^c(\mathsf{x}) \neq 0)}_I \\ * \mathsf{y} \mapsto 0,-,- * \mathsf{y} \Rrightarrow \pi,\emptyset * \mathsf{b} \doteq 0\end{array}\right\}$$

```
  }else{ x->c = y; b = true;
```
//Perform the action A^1_π

$$\left\{\pi^\bullet * \boxed{\begin{array}{l}\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \forall y \in \delta_1.\, \pi \notin \delta_1^p(y) \wedge \\ (\mathsf{x} \neq y \Rightarrow \pi \neq \delta_1^s(y)) \wedge \exists l,r.\, \delta_1(\mathsf{x}) = (\mathsf{y},\pi,-,l,r) \wedge \mathsf{y} \dot{\in} \delta_2 \wedge \mathsf{P}^{\pi.l}(l,\delta_1) \wedge \mathsf{P}^{\pi.r}(r,\delta_1))\end{array}}_I * \mathsf{b} \doteq 1\right\}$$

```
  }>
  if(b){ l = x->l; r = x->r;
```

$$\left\{\pi^\bullet * \boxed{\begin{array}{l}\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \forall y \in \delta_1.\, \pi \notin \delta_1^p(y) \wedge \\ (\mathsf{x} \neq y \Rightarrow \pi \neq \delta_1^s(y)) \wedge \delta_1(\mathsf{x}) = (\mathsf{y},\pi,-,\mathsf{l},\mathsf{r}) \wedge \mathsf{y} \dot{\in} \delta_2 \wedge \mathsf{P}^{\pi.l}(\mathsf{l},\delta_1) \wedge \mathsf{P}^{\pi.r}(\mathsf{r},\delta_1))\end{array}}_I\right\}$$

$$\left\{\begin{array}{l}\pi * \boxed{\begin{array}{l}\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \forall y \in \delta_1.\, \pi \notin \delta_1^p(y) \wedge \\ (\mathsf{x} \neq y \Rightarrow \pi \neq \delta_1^s(y)) \wedge \delta_1(\mathsf{x}) = (\mathsf{y},-,\pi,\mathsf{l},\mathsf{r}) \wedge \mathsf{y} \dot{\in} \delta_2)\end{array}}_I \\ * \pi.\mathsf{l}^\bullet * \boxed{\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \mathsf{P}^{\pi.l}(\mathsf{l},\delta_1))}_I \\ * \pi.\mathsf{r}^\bullet * \boxed{\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \mathsf{P}^{\pi.r}(\mathsf{r},\delta_1))}_I\end{array}\right\}$$

$$\left\{\pi * \boxed{\begin{array}{l}\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \forall y \in \delta_1.\, \pi \notin \delta_1^p(y) \\ \wedge (\mathsf{x} \neq y \Rightarrow \pi \neq \delta_1^s(y)) \wedge \delta_1(\mathsf{x}) = (\mathsf{y},-,\pi,\mathsf{l},\mathsf{r}) \wedge \mathsf{y} \dot{\in} \delta_2)\end{array}}_I \begin{array}{l} * \mathsf{Pre}(\mathsf{l},\pi.\mathsf{l},\delta) \\ * \mathsf{Pre}(\mathsf{r},\pi.\mathsf{r},\delta)\end{array}\right\}$$

$\{\mathsf{Pre}(\mathsf{l},\pi.\mathsf{l},\delta)\}$	$\{\mathsf{Pre}(\mathsf{r},\pi.\mathsf{r},\delta)\}$
`ll = copy_dag(l)`	`rr = copy_dag(r)`
$\{\mathsf{Post}(\mathsf{l},\mathsf{ll},\pi.\mathsf{l},\delta)\}$	$\{\mathsf{Post}(\mathsf{r},\mathsf{rr},\pi.\mathsf{r},\delta)\}$

$$\left\{\pi * \boxed{\begin{array}{l}\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \forall y \in \delta_1.\, \pi \notin \delta_1^p(y) \\ \wedge (\mathsf{x} \neq y \Rightarrow \pi \neq \delta_1^s(y)) \wedge \delta_1(\mathsf{x}) = (\mathsf{y},-,\pi,\mathsf{l},\mathsf{r}) \wedge \mathsf{y} \dot{\in} \delta_2)\end{array}}_I \begin{array}{l} * \mathsf{Post}(\mathsf{l},\mathsf{ll},\pi.\mathsf{l},\delta) \\ * \mathsf{Post}(\mathsf{r},\mathsf{rr},\pi.\mathsf{r},\delta)\end{array}\right\}$$

$$\left\{\pi^\bullet * \boxed{\begin{array}{l}\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \forall y \in \delta_1.\, \pi \notin \delta_1^p(y) \wedge \\ (\mathsf{x} \neq y \Rightarrow \pi \neq \delta_1^s(y)) \wedge \delta_1(\mathsf{x}) = (\mathsf{y},-,\pi,\mathsf{l},\mathsf{r}) \wedge \mathsf{y} \dot{\in} \delta_2 \wedge \mathsf{Q}^{\pi.l}(\mathsf{l},\mathsf{ll},\delta_1,\delta_2) \wedge \mathsf{Q}^{\pi.r}(\mathsf{r},\mathsf{rr},\delta_1,\delta_2))\end{array}}_I\right\}$$

```
    <y->l = ll>; <y->r = rr>;
```
//Perform A^2_π, A^3_π and A^4_π in order.

$$\left\{\pi^\bullet * \boxed{\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \mathsf{Q}^\pi(\mathsf{x},\mathsf{y},\delta_1,\delta_2))}_I\right\}$$

```
    return y;
```

$$\left\{\pi^\bullet * \boxed{\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \mathsf{Q}^\pi(\mathsf{x},\mathsf{ret},\delta_1,\delta_2))}_I\right\}$$

```
  }else{
```

$$\left\{\pi^\bullet * \boxed{\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \mathsf{Q}^\pi(\mathsf{x},\delta_1^c(\mathsf{x}),\delta_1,\delta_2) \wedge \delta_1^c(\mathsf{x}) \dot{\neq} 0)}_I \begin{array}{l} * \mathsf{y} \mapsto 0,-,- \\ * \mathsf{y} \Rrightarrow \pi,\emptyset\end{array}\right\}$$

```
    free(y, sizeof(struct node)) ; return x->c;
```

$$\left\{\pi^\bullet * \boxed{\exists\delta_1,\delta_2.\,\mathsf{icdag}(\delta_1,\delta_2) * (\delta \doteq \delta_1 \wedge \mathsf{Inv}(\delta_1,\delta_2) \wedge \mathsf{Q}^\pi(\mathsf{x},\mathsf{ret},\delta_1,\delta_2))}_I\right\}$$

```
} }
```

$$\{\mathsf{Post}(\mathsf{x},\mathsf{ret},\pi,\delta)\}$$

Changes reflected in the pure (mathematical) part as highlighted

The spatial part appears unchanged as highlighted

Verifying `copy_dag(x)`

```
struct node {struct node *c, *l, *r};
{Pre(x, π, δ)}
copy_dag(struct node *x) {struct node *l, *r, *ll, *rr, *y; bool b;
{π* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ P^π(x,δ1))) , }
  if(!x){ return 0; }
{π* * ret≐0 * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ Q^π(x,ret,δ1,δ2))) , }
  y = malloc(sizeof(struct node));
{π* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ P^π(x,δ1))) , * y ↦ 0,0,0 * y ⇒ π,∅ }
  <if(x->c){ b = false;        //Perform the action A^5_π
{π* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ Q^π(x,δ1^c(x),δ1,δ2) ∧ δ1^c(x) ≠ 0)) ,
 * y ↦ 0,−,− * y ⇒ π,∅ * b≐0 }
  }else{ x->c = y; b = true;        //Perform the action A^1_π
{π* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ ∀y∈δ1. π∉δ1^P(y) ∧
 (x≠y ⇒ π≠δ1^c(y)) ∧ ∃l,r. δ1(x)=(y,π,−,l,r) ∧ y∈δ2 ∧ P^{π.l}(l,δ1) ∧ P^{π.r}(r,δ1))) , * b≐1 }
  }>
  if(b){ l = x->l; r = x->r;
{π* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ ∀y∈δ1. π∉δ1^P(y) ∧
 (x≠y ⇒ π≠δ1^c(y)) ∧ δ1(x)=(y,π,−,l,r) ∧ y∈δ2 ∧ P^{π.l}(l,δ1) ∧ P^{π.r}(r,δ1))) , }
{π * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ ∀y∈δ1. π∉δ1^P(y) ∧
 (x≠y ⇒ π≠δ1^c(y)) ∧ δ1(x)=(y,−,π,l,r) ∧ y∈δ2)) ,
 * π.l* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ P^{π.l}(l,δ1))) ,
 * π.r* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ P^{π.r}(r,δ1))) , }
{π * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ ∀y∈δ1. π∉δ1^P(y)
 ∧ (x≠y ⇒ π≠δ1^c(y)) ∧ δ1(x)=(y,−,π,l,r) ∧ y∈δ2)) , * Pre(l,π.l,δ) * Pre(r,π.r,δ) }

        {Pre(l,π.l,δ)}        ||     {Pre(r,π.r,δ)}
        ll = copy_dag(l)      ||     rr = copy_dag(r)
        {Post(l,ll,π.l,δ)}    ||     {Post(r,rr,π.r,δ)}

{π * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ ∀y∈δ1. π∉δ1^P(y)
 ∧ (x≠y ⇒ π≠δ1^c(y)) ∧ δ1(x)=(y,−,π,l,r) ∧ y∈δ2)) , * Post(l,ll,π.l,δ) * Post(r,rr,π.r,δ) }
{π* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ ∀y∈δ1. π∉δ1^P(y) ∧
 (x≠y ⇒ π≠δ1^c(y)) ∧ δ1(x)=(y,−,π,l,r) ∧ y∈δ2 ∧ Q^{π.l}(l,ll,δ1,δ2) ∧ Q^{π.r}(r,rr,δ1,δ2))) , }
    <y->l = ll>; <y->r = rr>;  //Perform A^2_π, A^3_π and A^4_π in order.
{π* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ Q^π(x,y,δ1,δ2))) , }
    return y;  {π* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ Q^π(x,ret,δ1,δ2))) , }
  }else{
{π* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ Q^π(x,δ1^c(x),δ1,δ2) ∧ δ1^c(x) ≠ 0)) , * y ↦ 0,−,− * y ⇒ π,∅ }
    free(y, sizeof(struct node)) ; return x->c;
{π* * (∃δ1,δ2. icdag(δ1,δ2) * (δ≐δ1 ∧ Inv(δ1,δ2) ∧ Q^π(x,ret,δ1,δ2))) , }
} } {Post(x, ret, π, δ)}
```

icdag(δ_1, δ_2)

Changes reflected in the pure (mathematical) part as highlighted

The spatial part appears unchanged as highlighted

Conclusions

- ☑ Verified 4 concurrent fine-grained graph algorithms
  - ☑ Copying dags (directed acyclic graphs)
  - ☑ Speculative variant of Dijkstra's shortest path
  - ☑ Computing the spanning tree of a graph
  - ☑ Marking a graph
- ☑ Presented a common proof pattern for graph algorithms
  - ☑ *Abstract mathematical* graphs for Functional correctness
  - ☑ *Concrete Spatial (heap-represented)* graphs for memory safety
  - ☑ Combined reasoning for full proof
  - ☑ Inspired by existing logics where this pattern is "baked-in" to the model
  - ☑ "Baking-in" is unnecessary; demonstrated by CoLoSL reasoning

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Thank you for listening!

Speculative Concurrent Shortest Path

```
parallel_dijkstra((int[][] a, int[] c, int size, src) {
  bitarray work[size], done[size];
  for (i=0; i<size; i++){
    c[i] = a[src][i]; work[i] = 1; done[i] = 0;  //initialisation
  }; c[src] = 0;
  dijkstra(a,c,size,work,done) || ... || dijkstra(a,c,size,work,done)
}

dijkstra(int[][] a, int[] c, int size, bitarray work, done){
  i = 0;
  while(done != 2^size-1){
    b = <CAS(work[i], 1, 0)>;
    if(b){ cost = c[i];
      for(j=0; j<size; j++){ newcost = cost + a[i][j]; b = true;
        do{ oldcost = c[j];
          if(newcost < oldcost){
            b = <CAS(work[j], 1, 0)>;
            if(b){ b = <CAS(c[j], oldcost, newcost)>; <work[j] = 1>; }
            else { b = <CAS(done[j], 1, 0)>;
              if(b){ b = <CAS(c[j], oldcost, newcost)>;
                if(b){ < work[j] = 1 > } else { < done[j] = 1 > }
          } } }
        } while(!b)
      } < done[i] = 1 >;
    } i = (i+1) mod size;
} }
```