# Verifying Concurrent Graph Algorithms

**Azalea Raad**    Aquinas Hobor    Jules Villard    Philippa Gardner

Imperial College London
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Northern Concurrency Meeting
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# Concurrent Program Logic Genealogy

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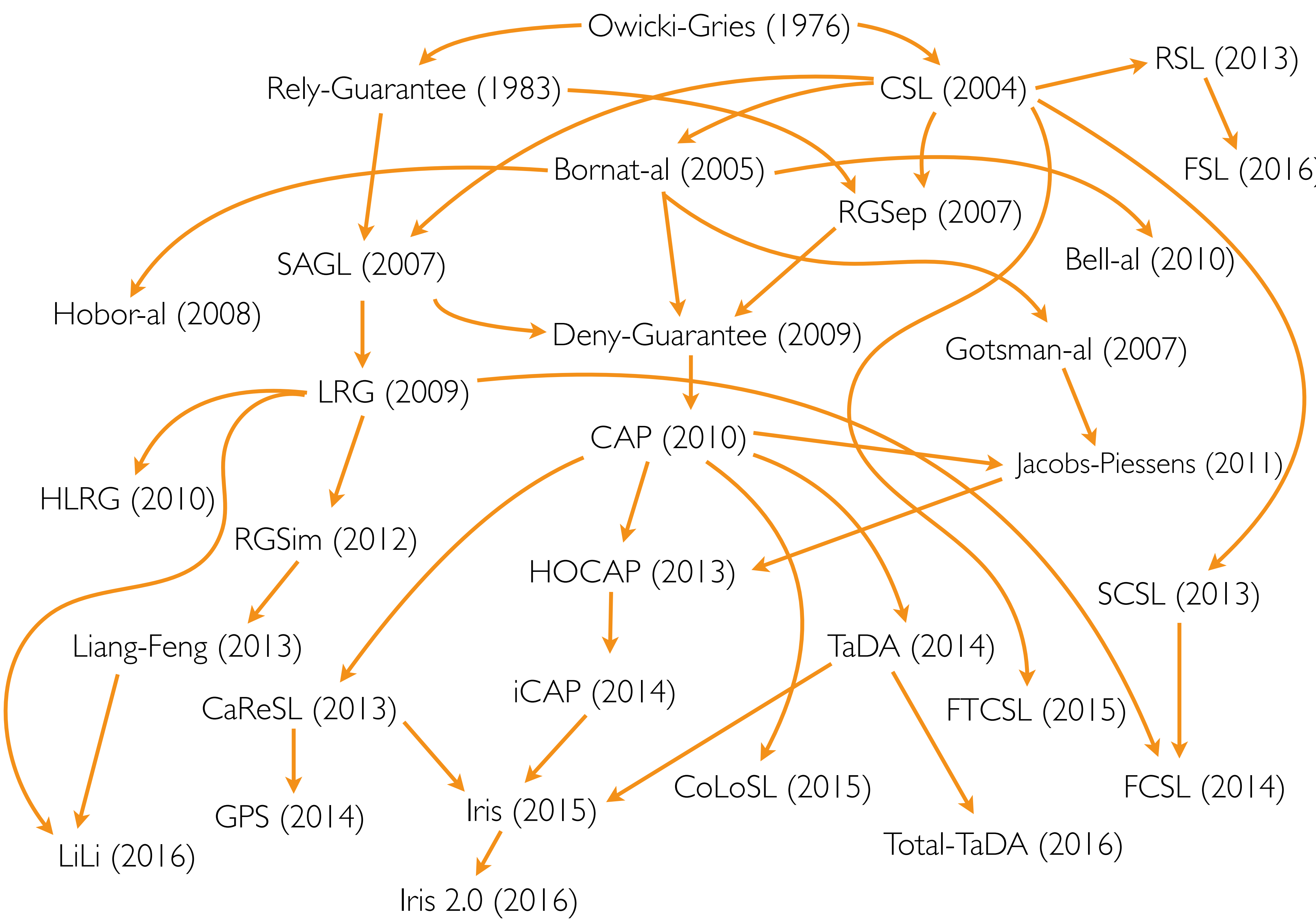


Graph credit: Ilya Sergey

# Graph Algorithms

Verifying **graph** algorithms is difficult…

- Subtle correctness argument
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**Local reasoning, separation and aliasing**


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**Abstract**

Structures built by pointer aliasing, such as DAGs and graphs, are notoriously tricky to deal with. The mechanisms of separation logic can deal with these structures, but so far this has been done by the maintenance of a global invariant. Specifications and proofs which use local reasoning, and which may point the way to a structured programming for pointers, are discussed. An idiom for inclusion – where one structure is included in another, is presented. A …


Pointers have been thought of as the gotos of data structuring, and so they can be, but if we can organise things so that our data structures and our programs come apart neatly in similar ways – structured reasoning for the heap – we shall perhaps be able to use pointers reliably. Separation logic facilitates local reasoning about parts of data structures: if an assignment or a procedure call alters only one corner of one data structure, then we can easily express and exploit the fact that it doesn't alter any other part of it nor any part of any separate data structure. The advantages can be seen most clearly in Yang's proof of the Schorr-Waite algorithm [12, 13], which elegantly avoids the struggles of [1] to show the same result using Hoare logic and array assignment. Apart from that work and Torp-Smith's proof of the Cheney algorithm [11], most work using separation logic has focussed on lists and trees, where separation is easily described.

In this paper we discuss three formal proofs (alas, not yet mechanically checked) of algorithms which deal with internal sharing of structure; two of the proofs deal with aliasing as well. Our aim has been to make the reasoning as local as possible. Although we can see considerable possibilities for further improvement, we can already claim significant success in extending the range of separation logic to new and difficult areas.

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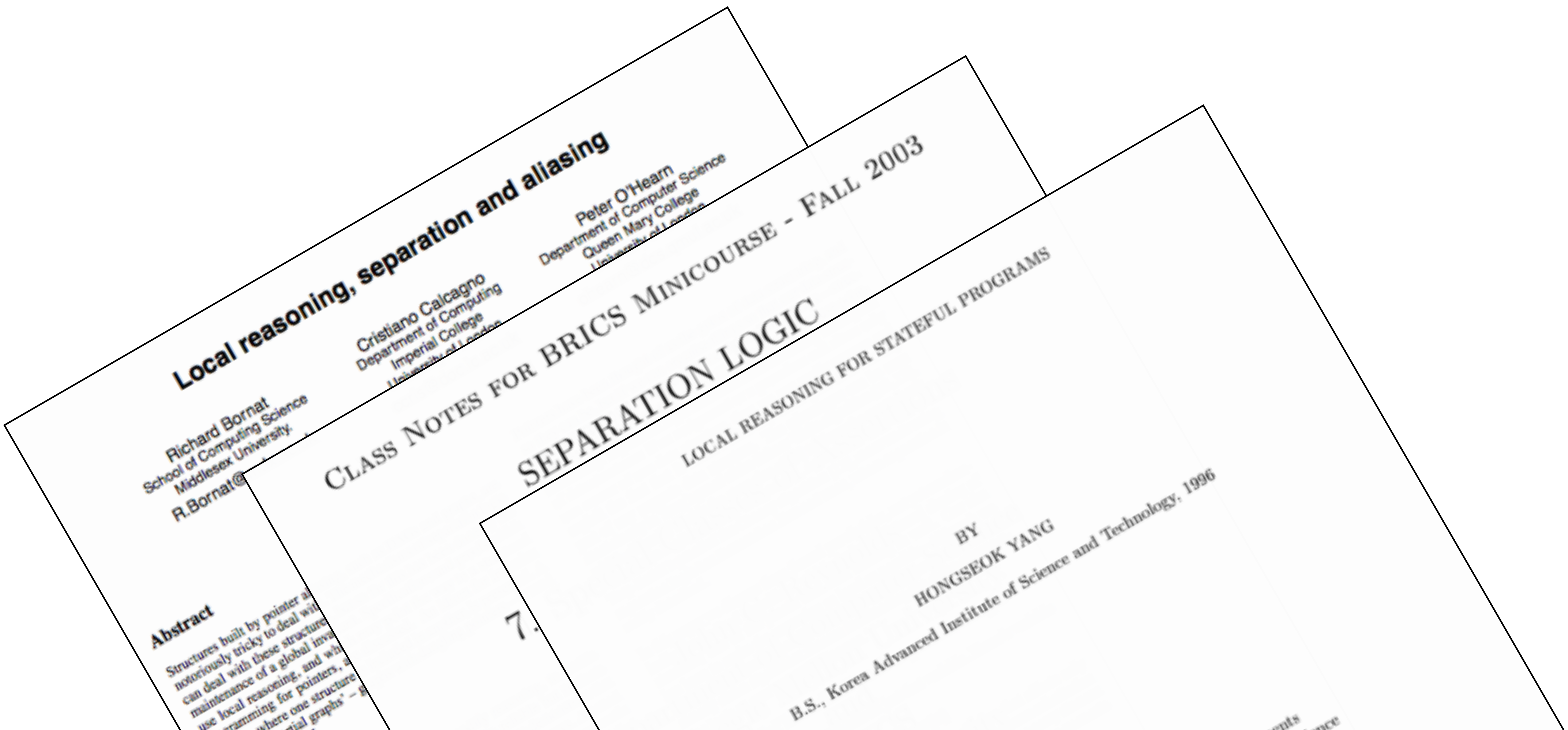

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# Concurrent Graph Algorithms

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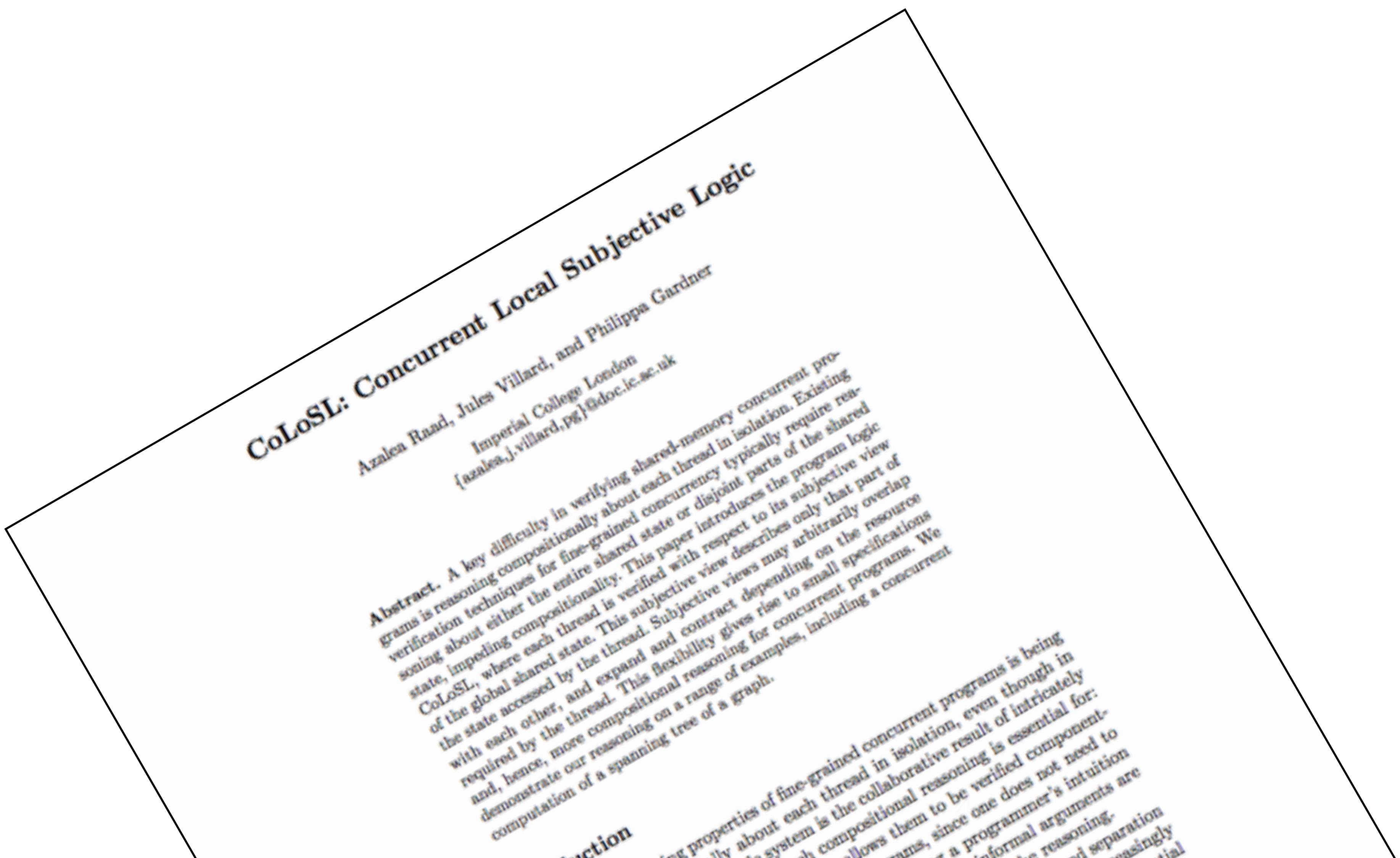

**CoLoSL: Concurrent Local Subjective Logic**

Azalea Raad, Jules Villard, and Philippa Gardner

Imperial College London
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**Abstract.** A key difficulty in verifying shared-memory concurrent programs is reasoning compositionally about each thread in isolation. Existing verification techniques for fine-grained concurrency typically require reasoning about either the entire shared state or disjoint parts of the shared state, impeding compositionality. This paper introduces the program logic CoLoSL, where each thread is verified with respect to its subjective view of the global shared state. This subjective view describes only that part of the state accessed by the thread. Subjective views may arbitrarily overlap with each other, and expand and contract depending on the resource required by the thread. This flexibility gives rise to small specifications and, hence, more compositional reasoning for concurrent programs. We demonstrate our reasoning on a range of examples, including a concurrent computation of a spanning tree of a graph.

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# Contributions

- Verified 4 concurrent fine-grained graph algorithms
  - Copying dags (directed acyclic graphs)
  - Speculative variant of Dijkstra's shortest path
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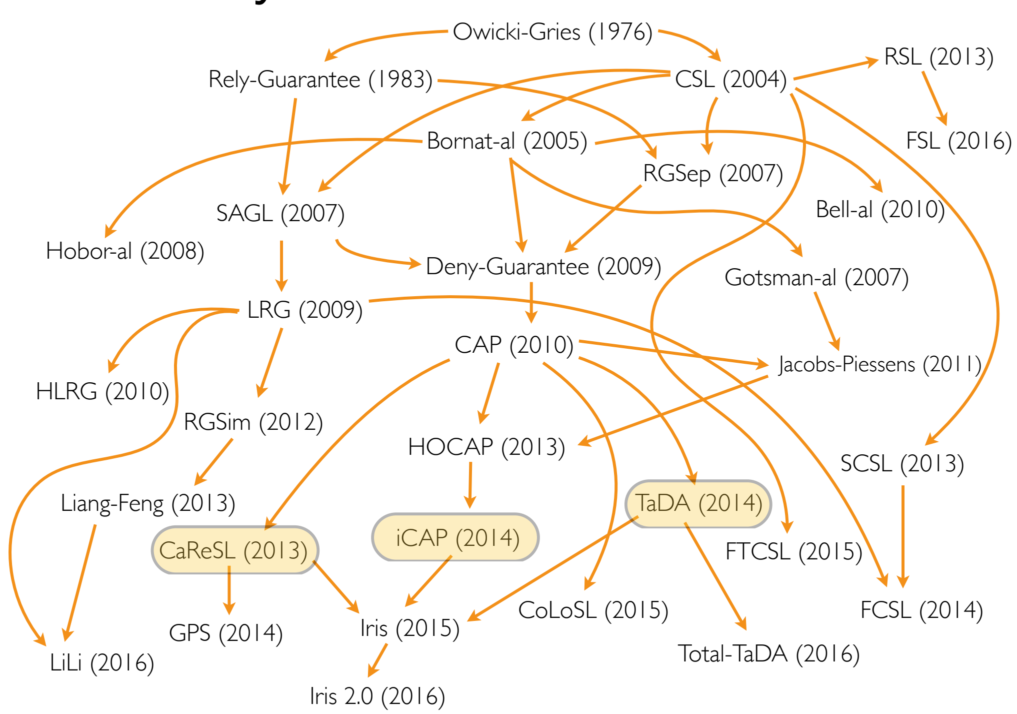

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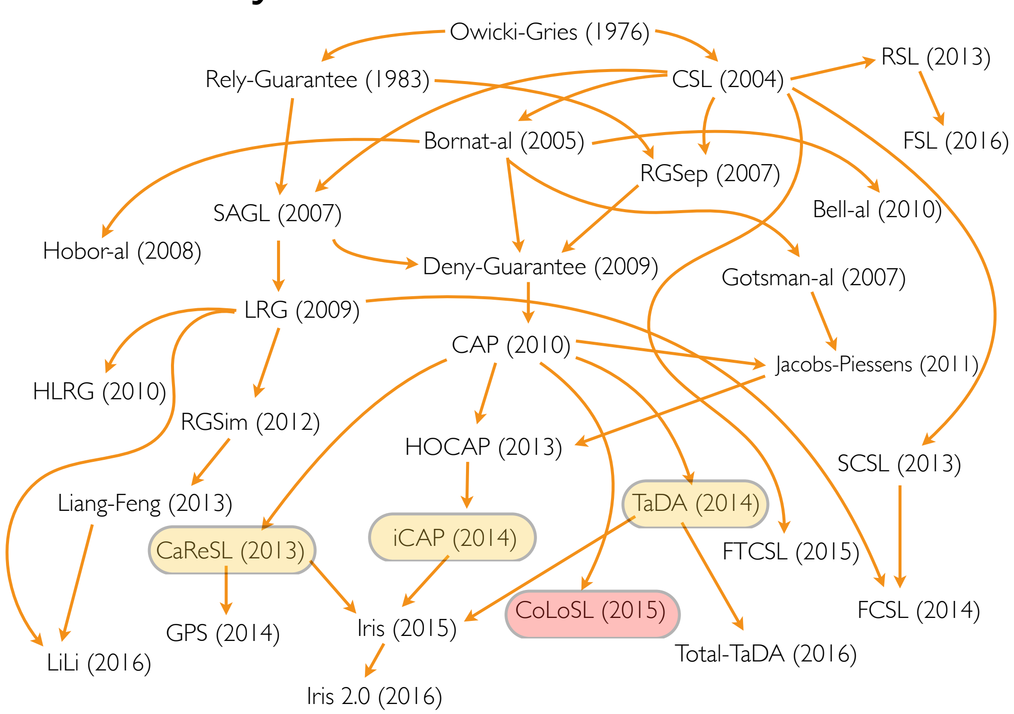

# This Talk

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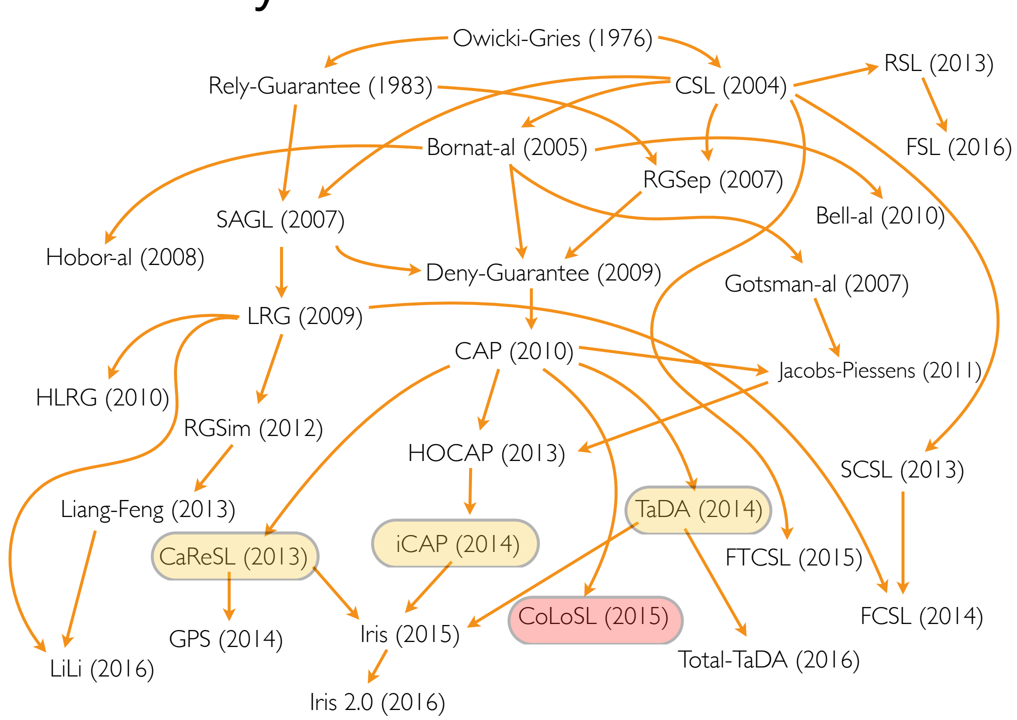

# Copying Binary DAGs

```
struct node {struct node *c, *l, *r}
copy_dag(struct node *x) {
  struct node *l, *r, *ll, *rr, *x';  bool b;
  if (!x) {return 0;}
  x' = malloc(sizeof(struct node));
  b = <CAS(x->c, 0, x')>;
  if (b) {
    l = x->l; r = x->r;
    ll = copy_dag(l)  ||  rr = copy_dag(r)
    <x'->l = ll>; <x'->r = rr>;
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  } else {
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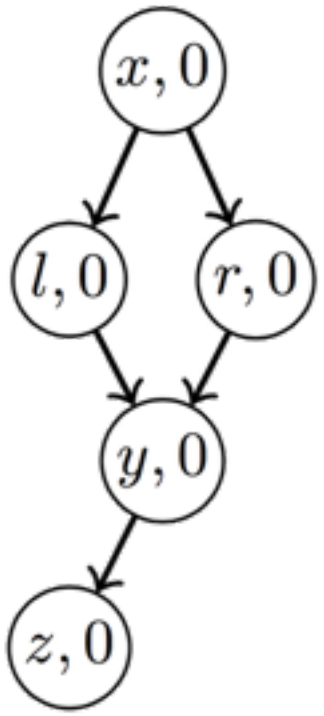
atomic blocks



# `copy_dag(x)` Specification

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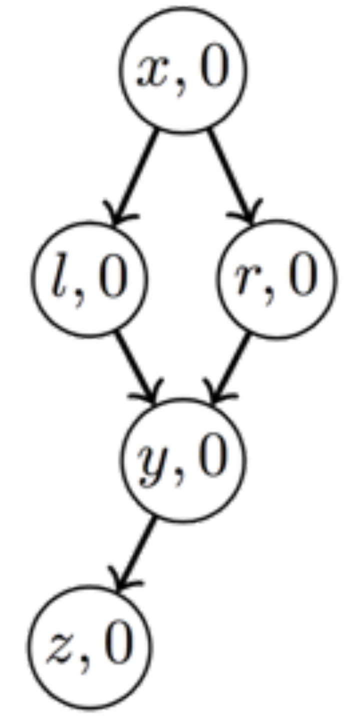


- Specification challenges
  - When **`copy_dag(x)`** returns, **`x`** is copied but its children may not be
  - If **`x`** is already copied, **`copy_dag(x)`** simply returns: the thread that copied **`x`** has made a *promise* to visit **`x`**'s children and ensure they are copied

# `copy_dag(x)`: A Trace

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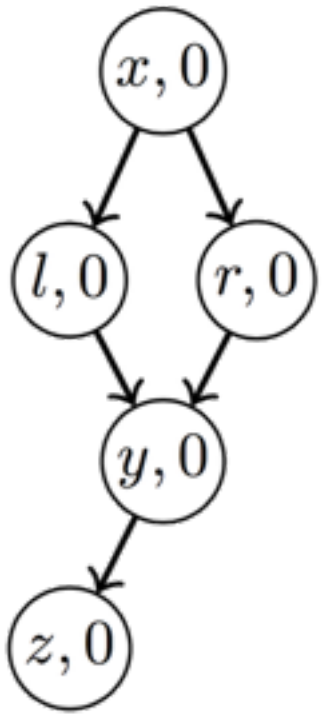

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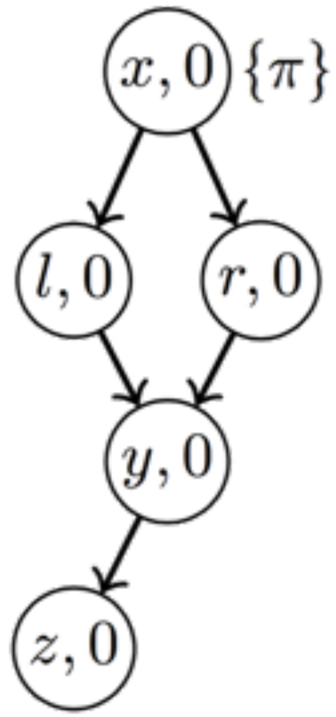

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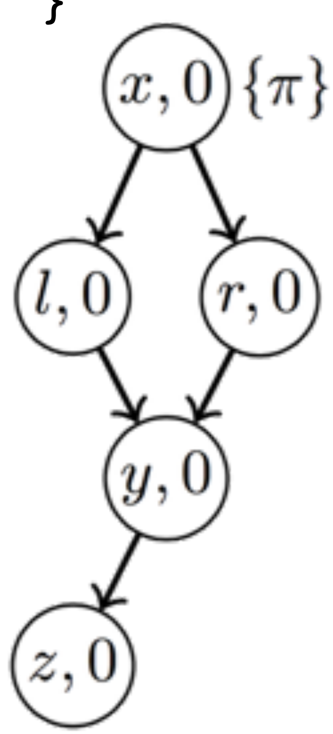


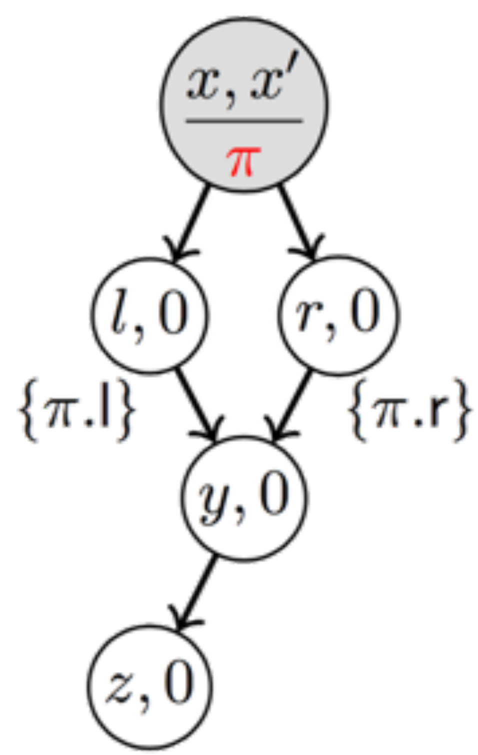

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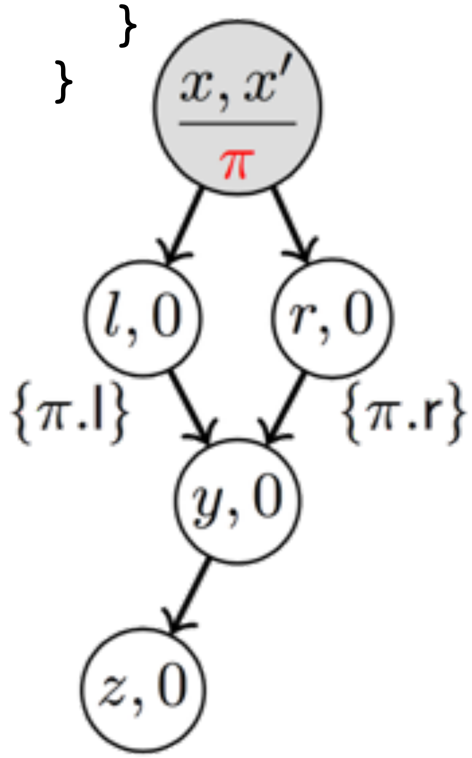


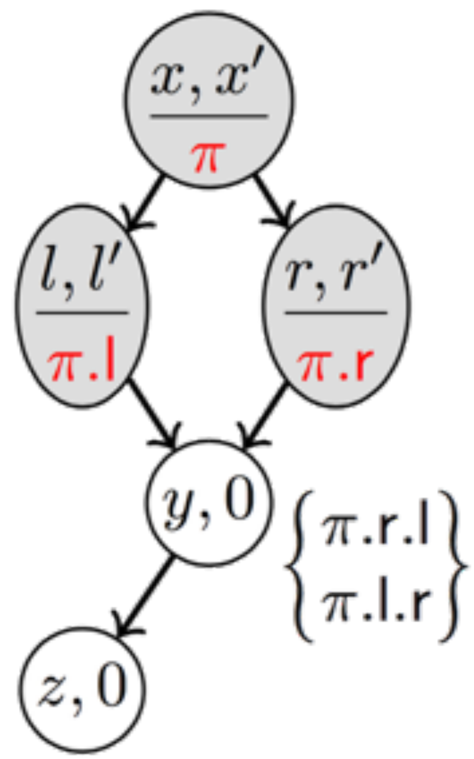

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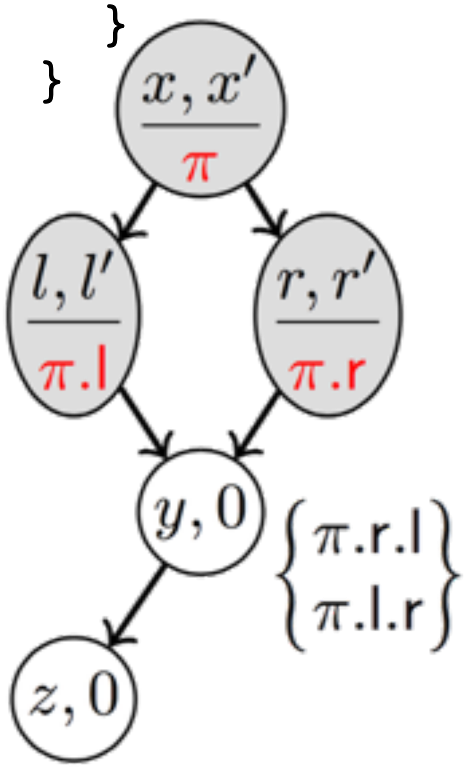


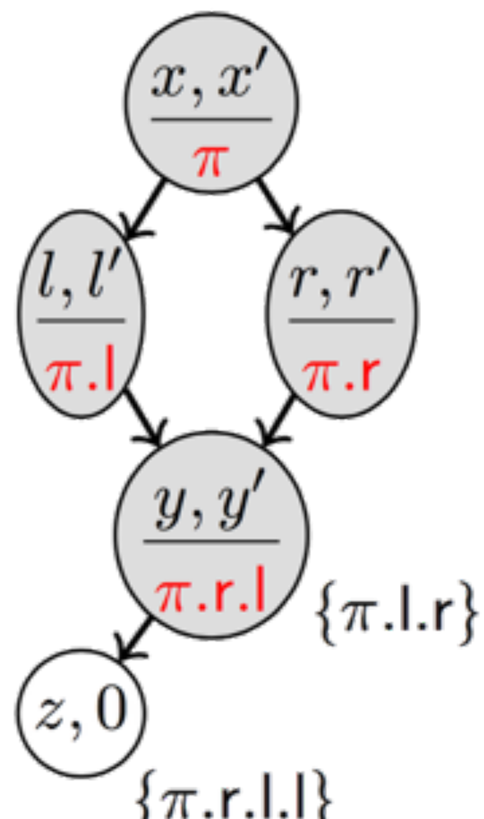

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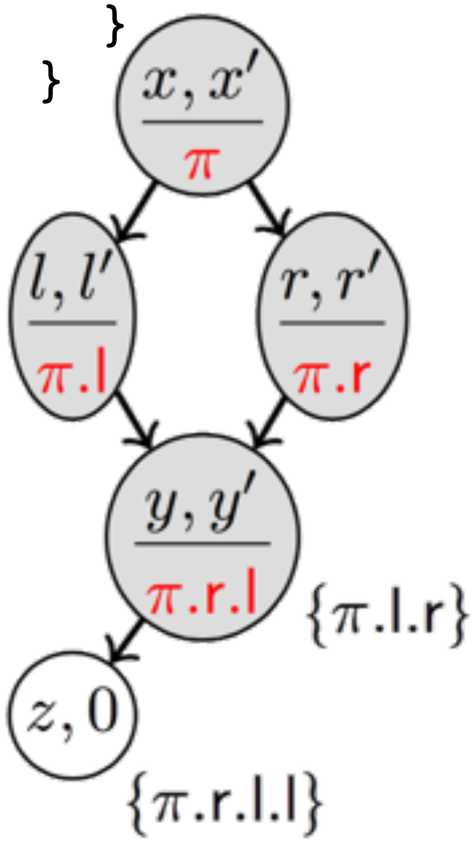


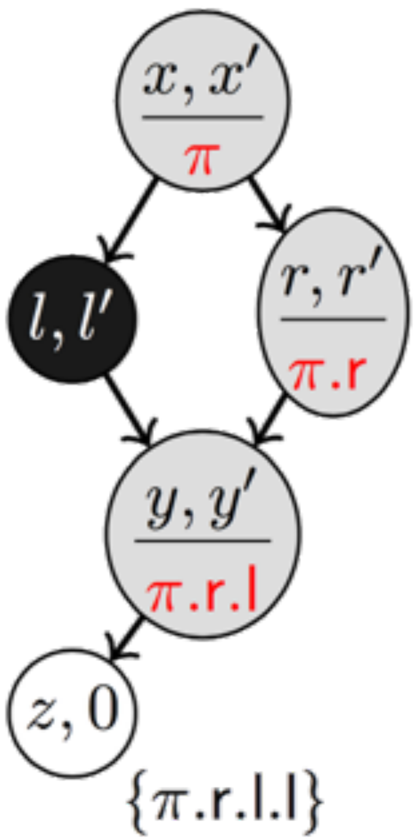

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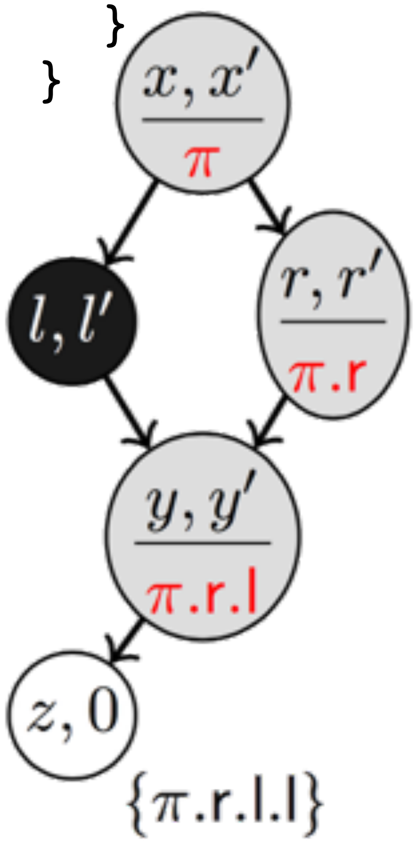


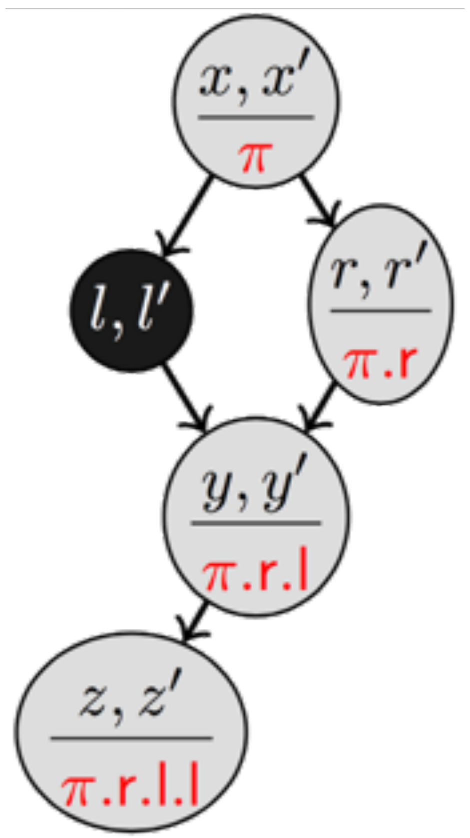

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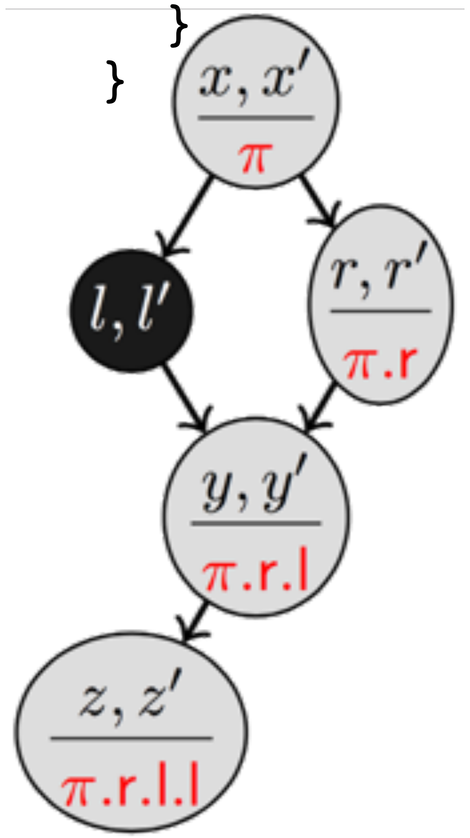


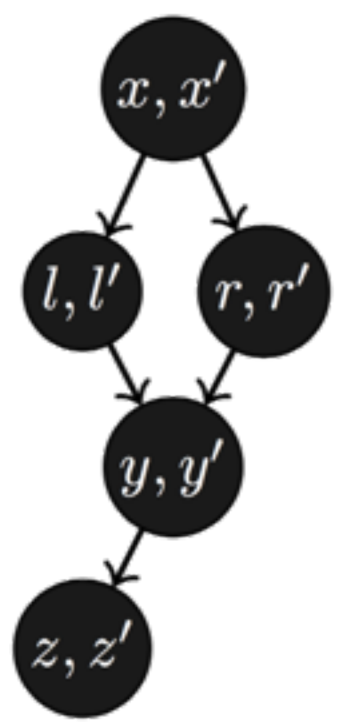

# 1. Tokens

A token mechanism for

- Thread identification
- Thread progress tracking

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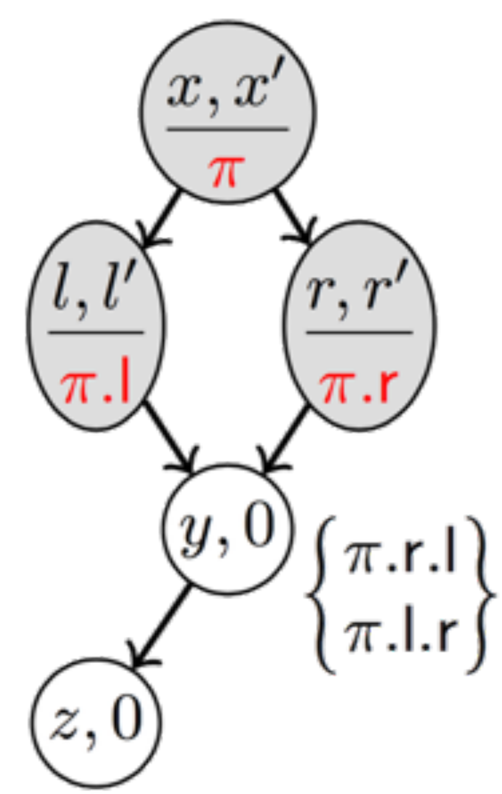
The `copy_dag` token mechanism for

- Thread identification
  - distinguish one token (thread) from another
  - identify two distinct sub-tokens given any token (at recursive call points)
  - model a parent-child relation (spawner-spawnee)
- Thread progress tracking
  - marking thread ids as tokens
  - promise sets as token sets



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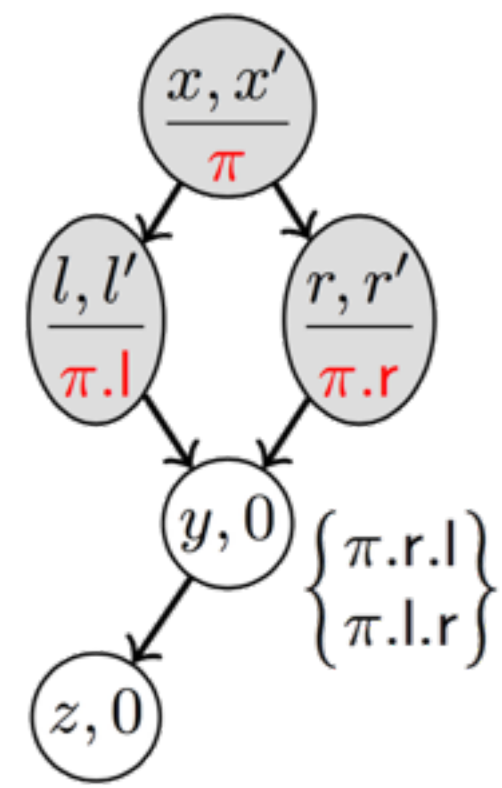

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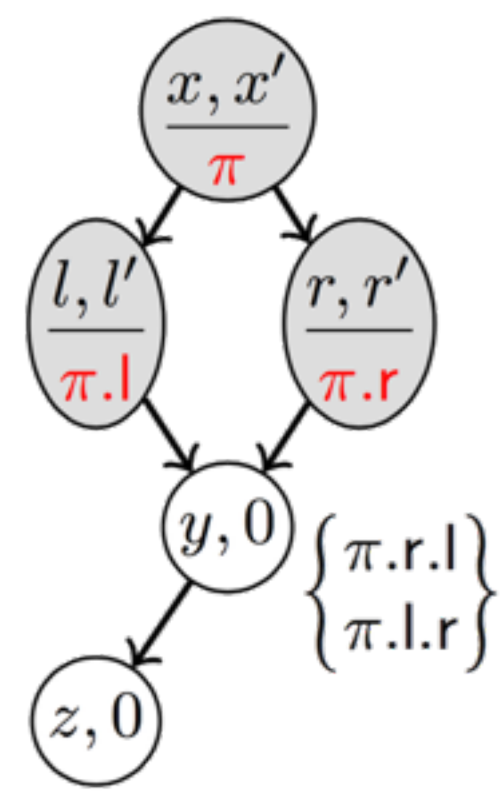

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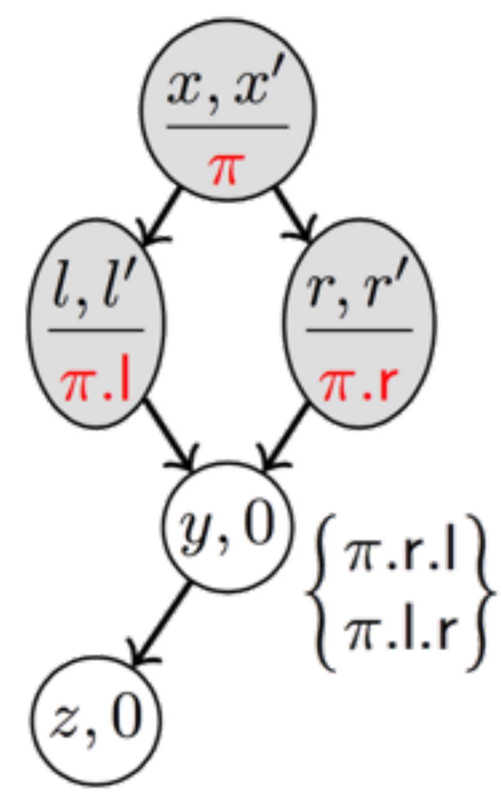

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top-most (maximal) token

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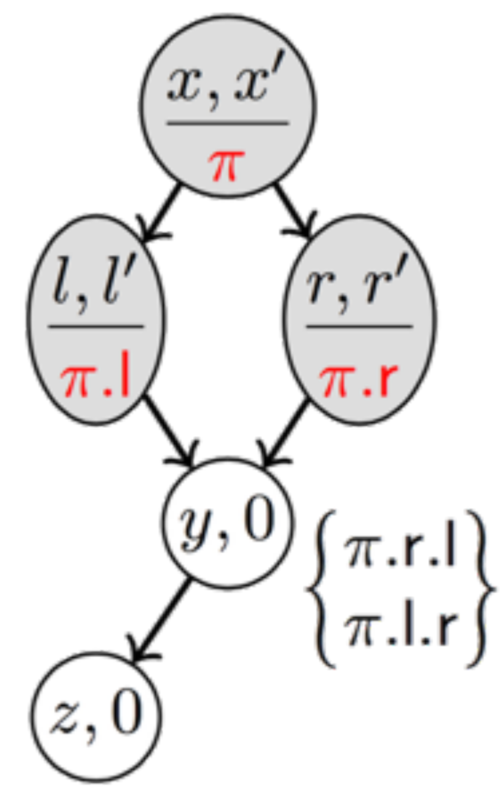
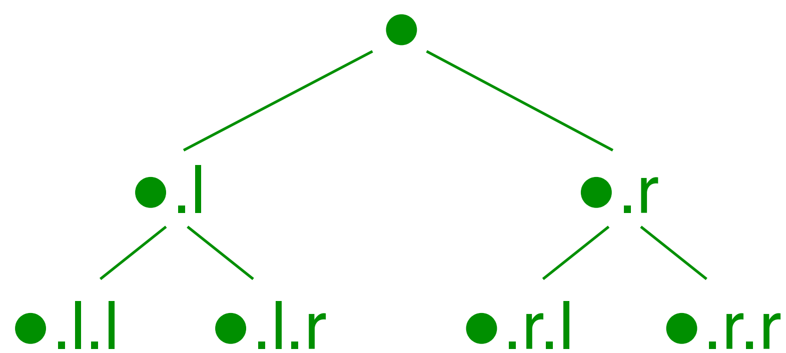
sub-thread relation



# 1. Tokens

The `copy_dag` token mechanism for

- Thread identification
  - distinguish one token (thread) from another
  - identify two distinct sub-tokens given any token (at recursive call points)
  - model a parent-child relation (spawner-spawnee)
- Thread progress tracking
  - marking thread ids as tokens
  - promise sets as token sets




top-most (maximal) token

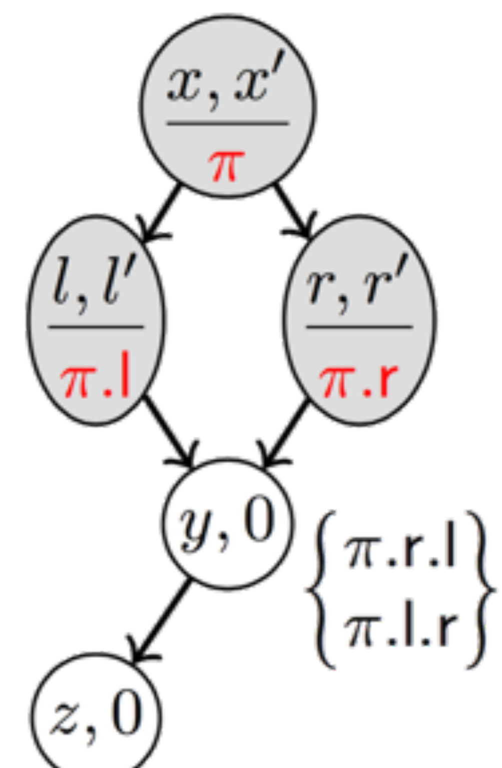
$$\pi ::= \bullet \mid \widehat{\circ\ \pi} \mid \widehat{\pi\ \circ}$$

$$\bullet.\mathsf{l} = \widehat{\bullet\ \circ} \qquad (\widehat{\circ\ \pi}).\mathsf{l} = \widehat{\circ\ \pi.\mathsf{l}} \qquad (\widehat{\pi\ \circ}).\mathsf{l} = \widehat{\pi.\mathsf{l}\ \circ}$$

$$\bullet.\mathsf{r} = \widehat{\circ\ \bullet} \qquad (\widehat{\circ\ \pi}).\mathsf{r} = \widehat{\circ\ \pi.\mathsf{r}} \qquad (\widehat{\pi\ \circ}).\mathsf{r} = \widehat{\pi.\mathsf{r}\ \circ}$$

$$\sqsubset\ = \{(\pi.\mathsf{l},\ \pi),\ (\pi.\mathsf{r},\ \pi)\}^{+}$$

sub-thread relation



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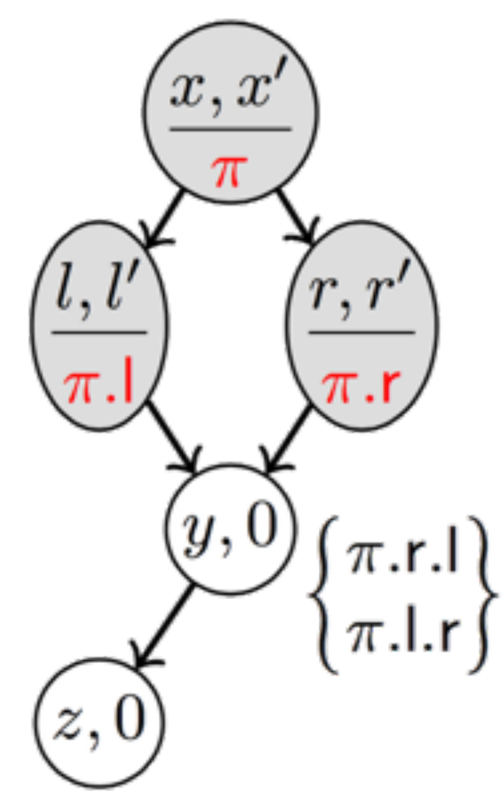
$$\delta = ( V , E , L )$$



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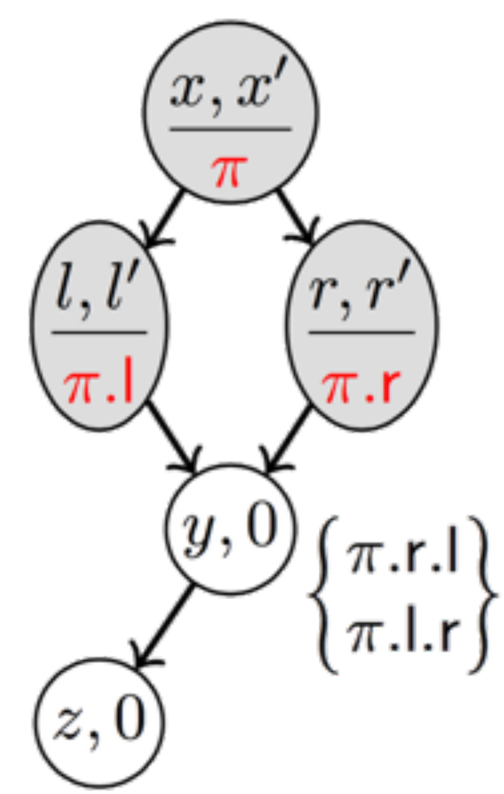
Vertices

$$\delta = (V, E, L)$$

$$V = \{x,\ l,\ r,\ y,\ z\}$$



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  - e.g. a pair of mathematical dags ($\delta$, $\delta_c$)
  - each dag is a triple:

Edges

$$\delta = ( V , E , L )$$

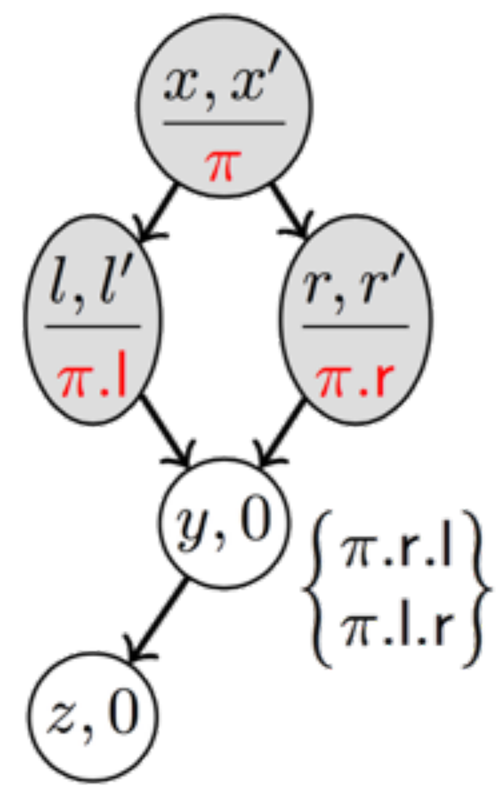
$$E(x) = l,\ r$$

$$E(l) = 0,\ y$$

$$E(r) = y,\ 0$$

$$E(y) = z,\ 0$$

$$E(z) = 0,\ 0$$



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  - e.g. a pair of mathematical dags ($\delta$, $\delta_c$)
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Labels

$$\delta = ( V , E , L )$$

copy

$$L\,(x) = x',\ \pi,\ \{\ \}$$

$$L\,(l) = l',\ \pi.\text{l},\ \{\ \}$$

$$L\,(r) = r',\ \pi.\text{r},\ \{\ \}$$

$$L\,(y) = 0,\ 0,\ \{\pi.\text{r.l},\ \pi.\text{l.r}\}$$

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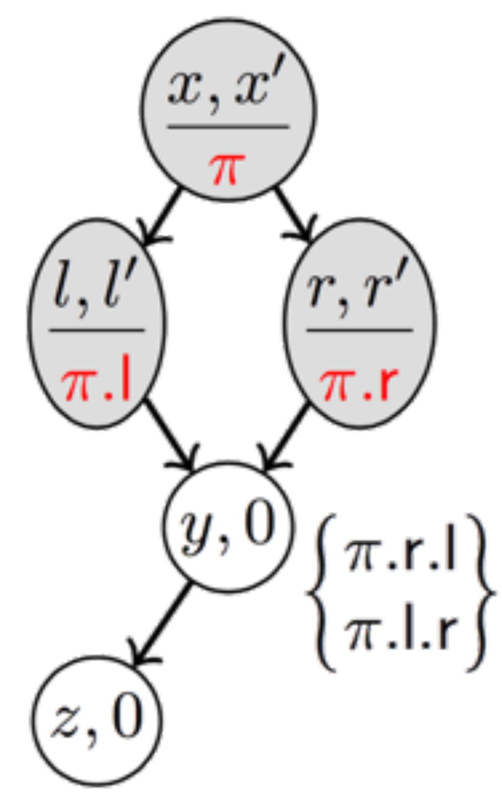

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Labels

$$\delta = (V, E, L)$$

copy    copying thread

$$L(x) = x', \pi, \{\}$$

$$L(l) = l', \pi.\text{l}, \{\}$$

$$L(r) = r', \pi.\text{r}, \{\}$$

$$L(y) = 0, 0, \{\pi.\text{r.l}, \pi.\text{l.r}\}$$

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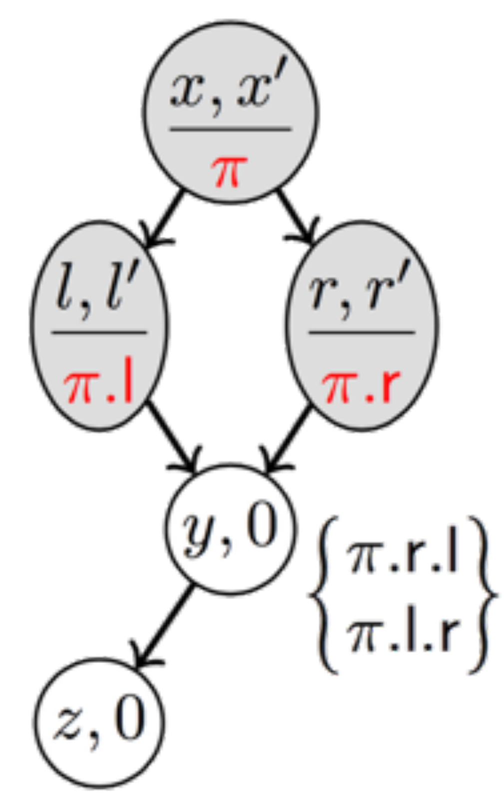

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copy　copying thread　promise set

$L(x) = x', \pi, \{\ \}$

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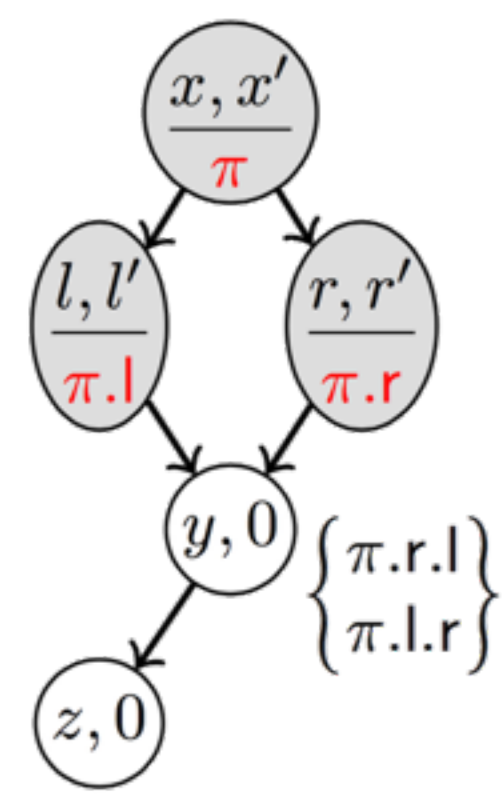

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Labels

$$\delta = (V, E, L)$$

copy  copying thread  promise set

ghost components

$$L(x) = x', \pi, \{\}$$

$$L(l) = l', \pi.\mathrm{l}, \{\}$$

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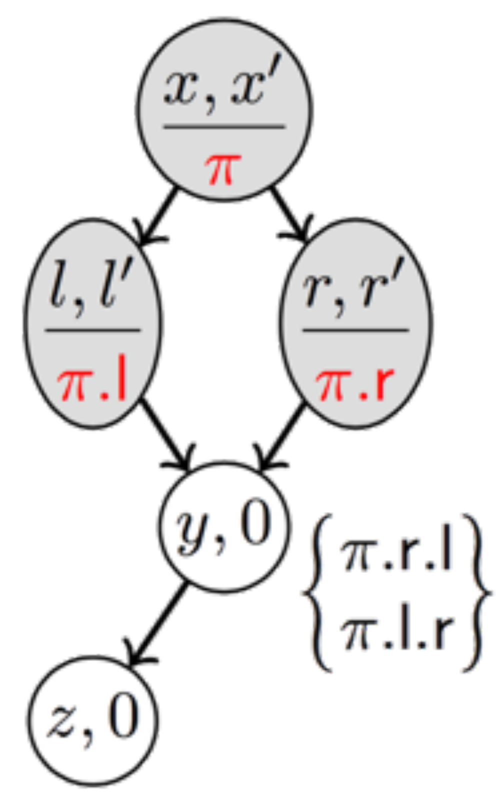

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Labels

$$\delta = (V, E, L)$$

copy($x$) thread($x$) promise($x$)

$$L(x) = x', \ \pi, \ \{\ \}$$

$$L(l) = l', \ \pi.\mathrm{l}, \ \{\ \}$$

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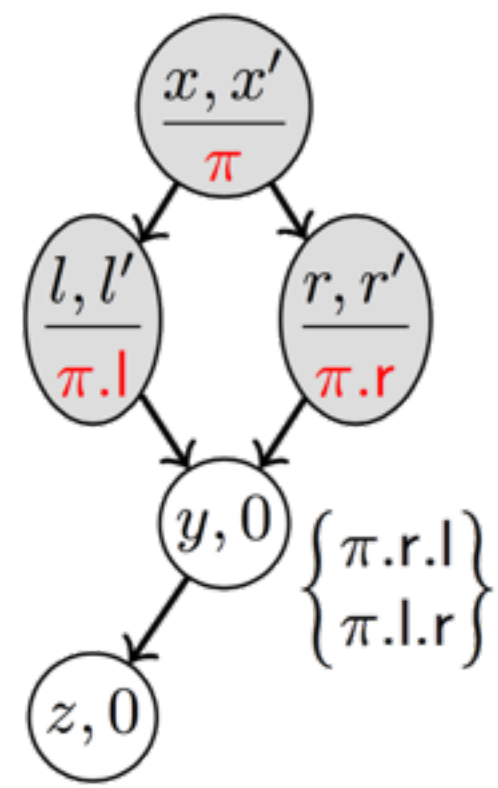

# 3. Mathematical Actions

- An abstraction of thread actions (on mathematical objects)
  - atomic blocks as well as ghost actions
  - $A^\pi$ denotes the actions of thread $\pi$

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  - $A^{\pi}$ denotes the actions of thread $\pi$

```
struct node {struct node *c, *l, *r}
copy_dag(struct node *x) {
  struct node *l, *r, *ll, *rr, *x';  bool b;
  if (!x) {return 0;}
  x' = malloc(sizeof(struct node));
  b = <CAS(x->c, 0, x')>;
  if (b) {
    l = x->l; r = x->r;
    ll = copy_dag(l)  ||  rr = copy_dag(r)
    <x'->l = ll>; <x'->r = rr>;
    return x';
  } else {
    free(x', sizeof(struct node));  return x->c;
  }
}
```

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```
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copy_dag(
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  x' =
  b = <CAS(x->c, 0, x')>;
  if
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    <x'->l = ll>; <x'->r = rr>

  } else {

  }
}
```

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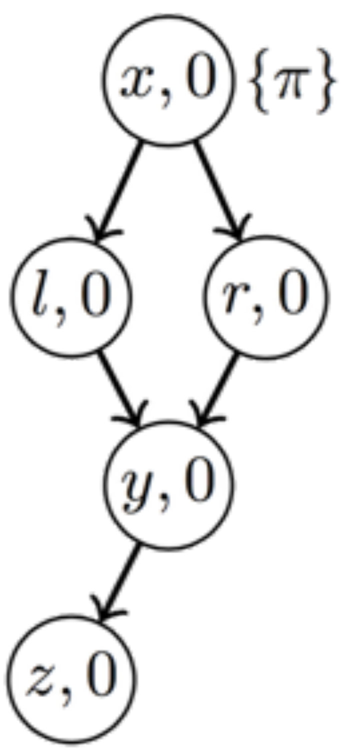


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b = <CAS(x->c, 0, x')>;
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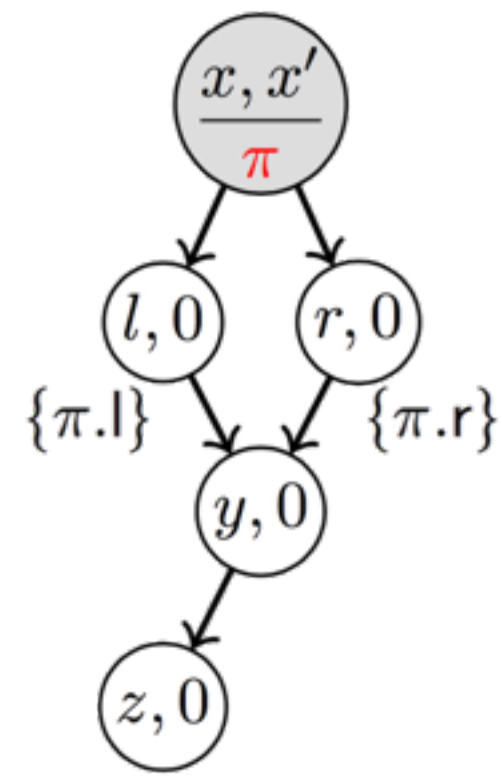


$(\delta, \delta_c) = ((V, E, L), (V_c, E_c, L_c))$ $\overset{A^{\pi}}{\rightsquigarrow}$ $(\delta', \delta'_c) = ((V, E, L'), (V'_c, E'_c, L'_c))$

$L(x) = 0,\ 0,\ \{\pi\}$
$L(l) = 0,\ 0,\ \{\ \}$
$L(r) = 0,\ 0,\ \{\ \}$

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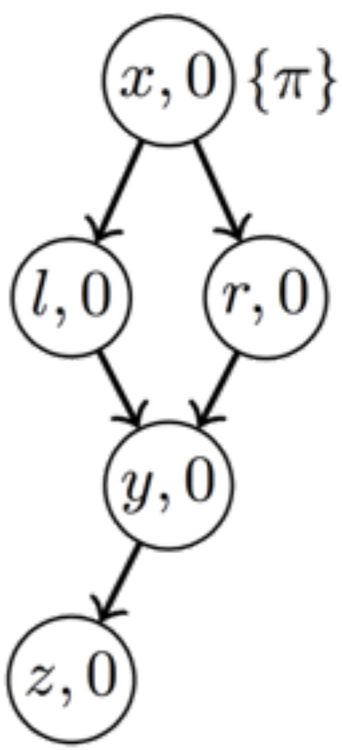


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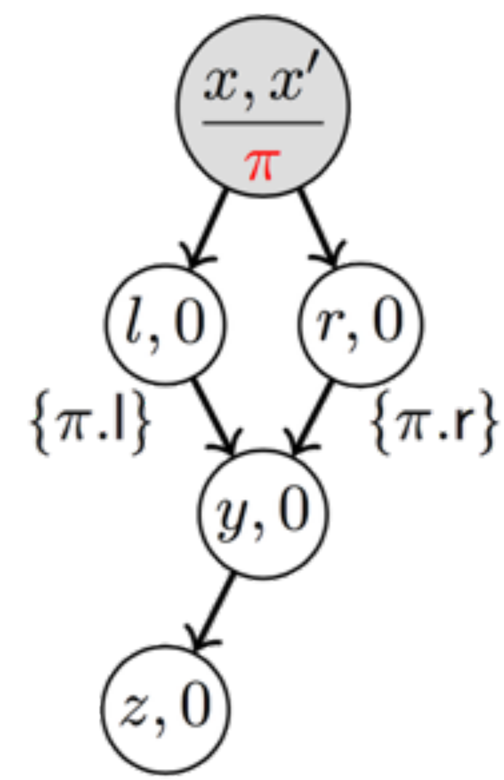
struct
copy_dag(
  struct
  if
  x' =
  b = <CAS(x->c, 0, x')>;
  if
    l = x->l; r = x->r;
    ll = copy_dag(l)  ||  rr = copy_dag(r)
    <x'->l = ll>; <x'->r = rr>

  } else {

  }
}
```




$(\delta, \delta_C) = ((V, E, L), (V_C, E_C, L_C))$ $\overset{A^{\pi}}{\rightsquigarrow}$ $(\delta', \delta'_C) = ((V, E, L'), (V'_C, E'_C, L'_C))$

$L(x) = 0,\ 0,\ \{\pi\}$
$L(l) = 0,\ 0,\ \{\ \}$
$L(r) = 0,\ 0,\ \{\ \}$

$L' = L[x \mapsto x',\ \pi,\ \{\ \}][l \mapsto 0, 0, \{\pi.\mathrm{l}\}][r \mapsto 0, 0, \{\pi.\mathrm{r}\}]$

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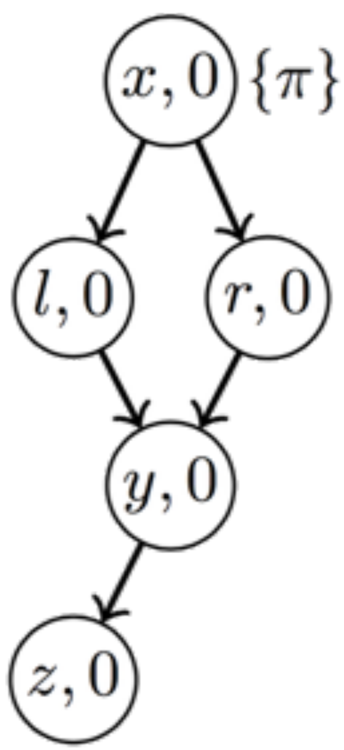


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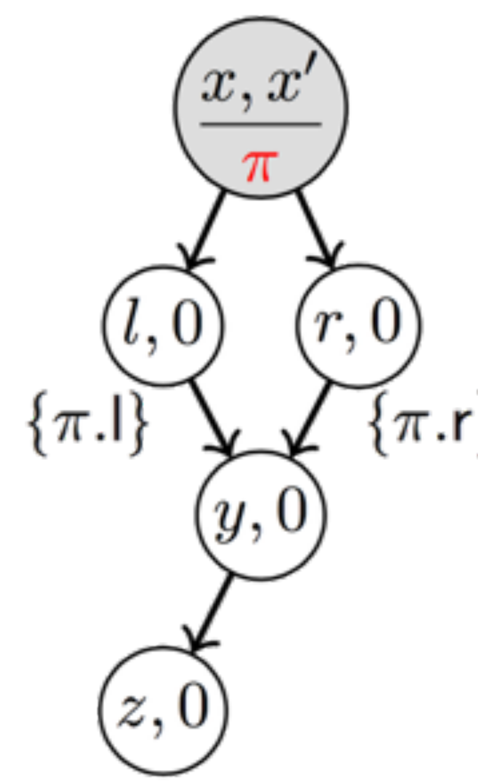


$(\delta, \delta_C) = ( (V, E, L), (V_C, E_C, L_C) )$ $\overset{A^{\pi}}{\rightsquigarrow}$ $(\delta', \delta'_C) = ( (V, E, L'), (V'_C, E'_C, L'_C) )$

$L(x) = 0,\ 0,\ \{\pi\}$
$L(l) = 0,\ 0,\ \{\ \}$
$L(r) = 0,\ 0,\ \{\ \}$

$L' = L[x \mapsto x',\ \pi,\ \{\ \}][l \mapsto 0, 0, \{\pi.\mathsf{l}\}][r \mapsto 0, 0, \{\pi.\mathsf{r}\}]$

$V'_C = V_C \uplus \{x'\}$ $\quad E'_C = E_C \uplus [x' \mapsto \cdots]$ $\quad L'_C = L_C \uplus [x' \mapsto \cdots]$

# 4. Mathematical Specification

Inv(δ, $\delta_c$) ≜ δ and $\delta_c$ are both acyclic;
every node $x$' in the copy $\delta_c$ corresponds to a unique node $x$ in the source δ;
every node $x$ in the source δ has some copy value $x$'

# 4. Mathematical Specification

$\mathrm{Inv}(\delta, \delta_c) \triangleq \mathrm{acyc}(\delta) \wedge \mathrm{acyc}(\delta_c) \wedge$

every node $x'$ in the copy $\delta_c$ corresponds to a unique node $x$ in the source $\delta$;
every node $x$ in the source $\delta$ has some copy value $x'$

# 4. Mathematical Specification

Inv(δ, $\delta_c$) ≜ acyc(δ) ∧ acyc($\delta_c$) ∧ $\forall x' \in \delta_c.\ \exists! x \in \delta.$ copy($x$)= $x'$ ∧

every node $x$ in the source δ has some copy value $x'$

# 4. Mathematical Specification

$\mathrm{Inv}(\delta, \delta_c) \triangleq \mathrm{acyc}(\delta) \land \mathrm{acyc}(\delta_c) \land \forall x' \in \delta_c.\ \exists! x \in \delta.\ \mathrm{copy}(x) = x'$
$\quad \land \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \land \mathrm{ic}(x, x', \delta, \delta_c)$

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$\mathrm{Inv}(\delta, \delta_c) \triangleq \mathrm{acyc}(\delta) \wedge \mathrm{acyc}(\delta_c) \wedge \forall x' \in \delta_c.\ \exists! x \in \delta.\ \mathrm{copy}(x) = x'$
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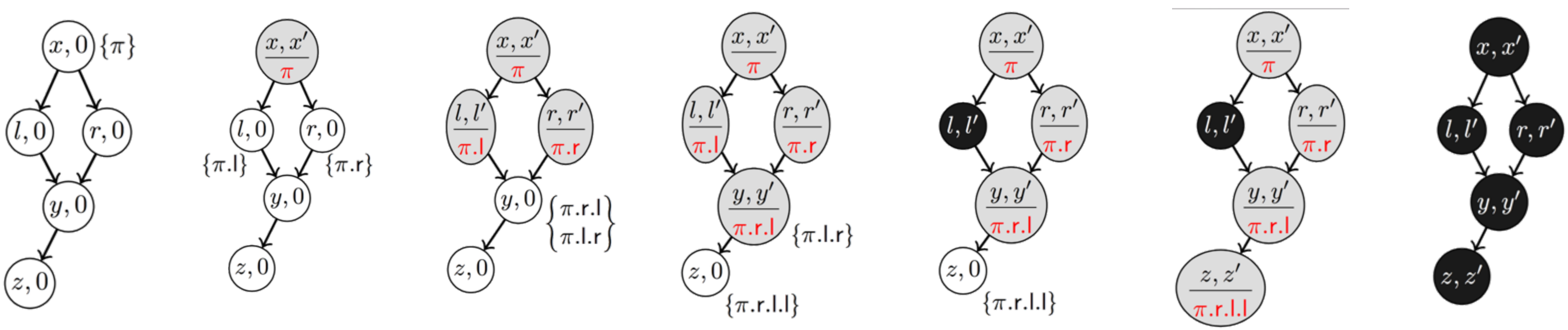
$\mathrm{ic}(x, x', \delta, \delta_c) \triangleq$ if $x'$ is 0 ($x$ is not copied yet), then $x$ will eventually be copied:

# 4. Mathematical Specification

Inv(δ, $\delta_c$) ≜ acyc(δ) ∧ acyc($\delta_c$) ∧ ∀$x$' ∈ $\delta_c$. ∃!$x$ ∈ δ. copy($x$)= $x$'
∧ ∀$x$ ∈ δ. ∃$x$'. copy($x$)=$x$' ∧ ic($x$, $x$', δ, $\delta_c$)

ic($x$, $x$', δ, $\delta_c$) ≜ if $x$' is 0 ($x$ is not copied yet), then $x$ will eventually be copied:

there exists some $y$ in δ s.t.
1) the promise set of $y$ is non-empty; 2) $y$ can reach $x$ along a path p;
and 3) every node along the path p is not copied
⇒ when $y$ is eventually copied, it'll visit $x$ along p and copy it too



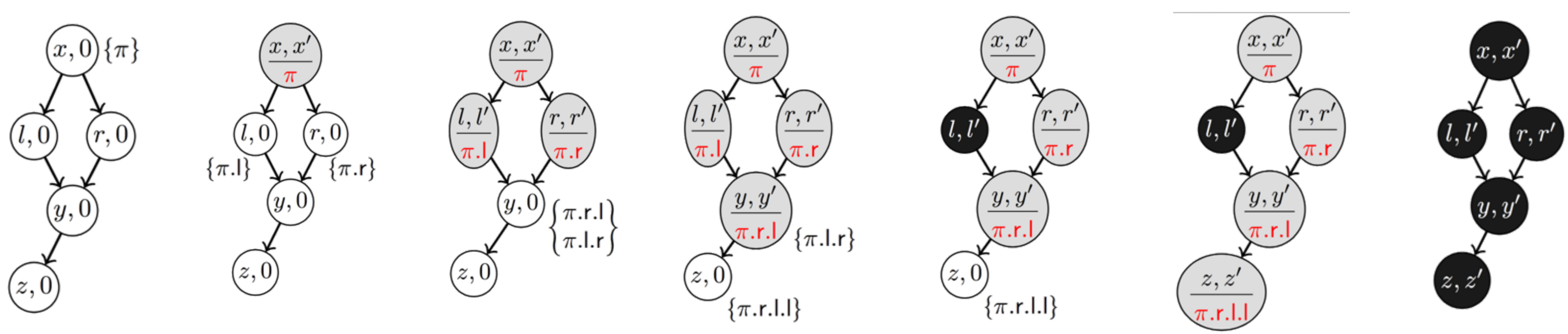
# 4. Mathematical Specification

Inv(δ, $\delta_c$) ≜ acyc(δ) ∧ acyc($\delta_c$) ∧ $\forall x' \in \delta_c.\ \exists! x \in \delta.$ copy($x$)= $x'$
∧ $\forall x \in \delta.\ \exists x'.$ copy($x$)=$x'$ ∧ ic($x$, $x'$, δ, $\delta_c$)

ic($x$, $x'$, δ, $\delta_c$) ≜ if $x'$ is 0 ($x$ is not copied yet), then $x$ will eventually be copied:

there exists some $y$ in δ s.t.
1) the promise set of $y$ is non-empty; 2) $y$ can reach $x$ along a path p; and 3) every node along the path p is not copied

otherwise, $x'$ is a node in $\delta_c$ and
the children of $x$, ($l$,$r$), are also copied to some ($l'$,$r'$):
ic($l$, $l'$, δ, $\delta_c$) and ic($r$, $r'$, δ, $\delta_c$)



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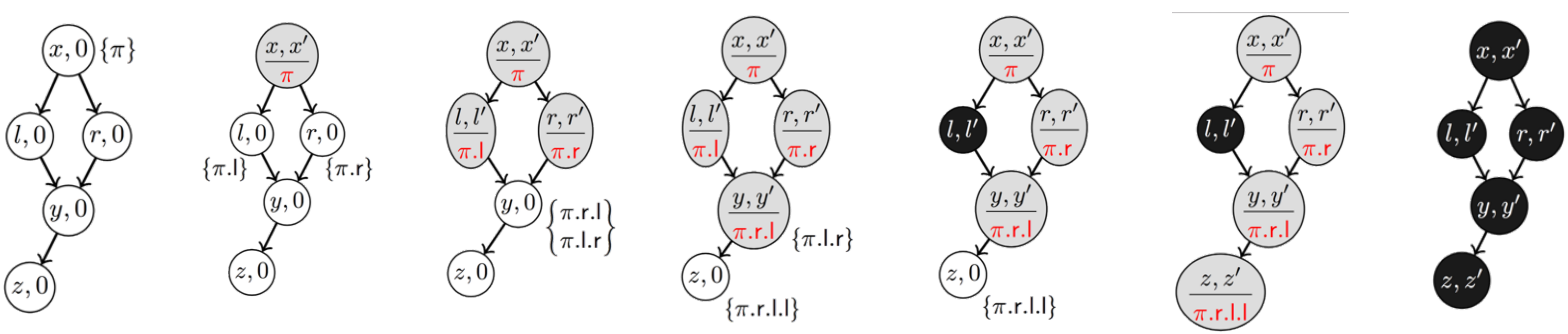
$\mathsf{Inv}(\delta, \delta_c) \triangleq \mathsf{acyc}(\delta) \wedge \mathsf{acyc}(\delta_c) \wedge \forall x' \in \delta_c.\ \exists! x \in \delta.\ \mathsf{copy}(x) = x'$
$\quad \wedge\ \forall x \in \delta.\ \exists x'.\ \mathsf{copy}(x) = x' \wedge \mathsf{ic}(x, x', \delta, \delta_c)$

$$
\begin{aligned}
\mathsf{ic}(x, x', \delta, \delta_c) \triangleq\ & (x{=}0 \wedge x'{=}0) \vee \\
& \Big(x{\neq}0 \wedge \Big[(x'{=}0 \wedge \delta^{c}(x){=}x' \wedge \exists y.\ \delta^{p}(y){\neq}\emptyset \wedge y \overset{\delta}{\rightsquigarrow}{}^{*}_{0}\ x) \\
& \quad \vee (x'{\neq}0 \wedge x' \in \delta' \wedge \exists \pi, l, r, l', r'.\ \delta(x){=}((x', \pi, -), l, r) \wedge \delta'(x'){=}(-, l', r') \\
& \qquad \wedge (l'{\neq}0 \Rightarrow \mathsf{ic}(l, l', \delta, \delta')) \wedge (r'{\neq}0 \Rightarrow \mathsf{ic}(r, r', \delta, \delta'))) \\
& \quad \vee (x'{\neq}0 \wedge x' \in \delta' \wedge \exists l, r, l', r'.\ \delta(x){=}((x', 0, -), l, r) \wedge \delta'(x'){=}(-, l', r') \\
& \qquad \wedge \mathsf{ic}(l, l', \delta, \delta') \wedge \mathsf{ic}(r, r', \delta, \delta'))\Big]\Big)
\end{aligned}
$$



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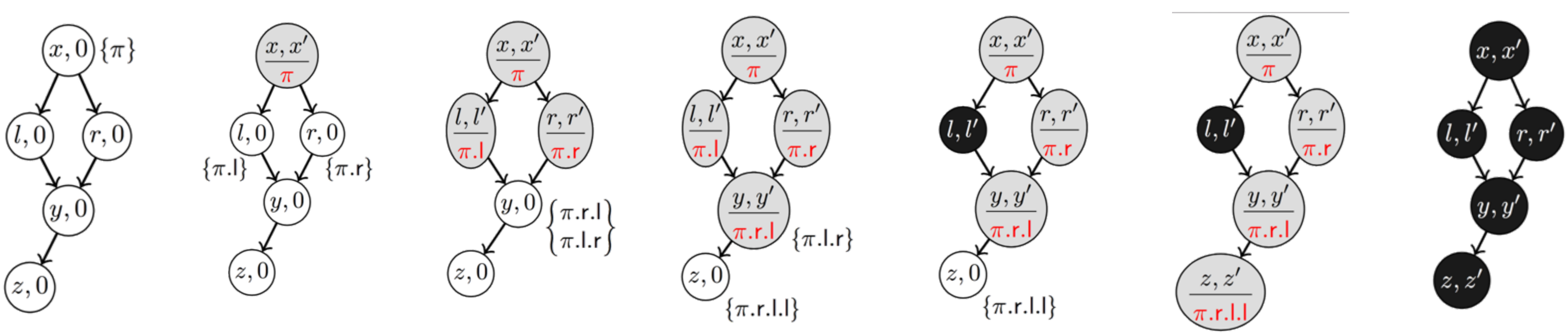
$\wedge\ \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \wedge \mathrm{ic}(x, x', \delta, \delta_c)$

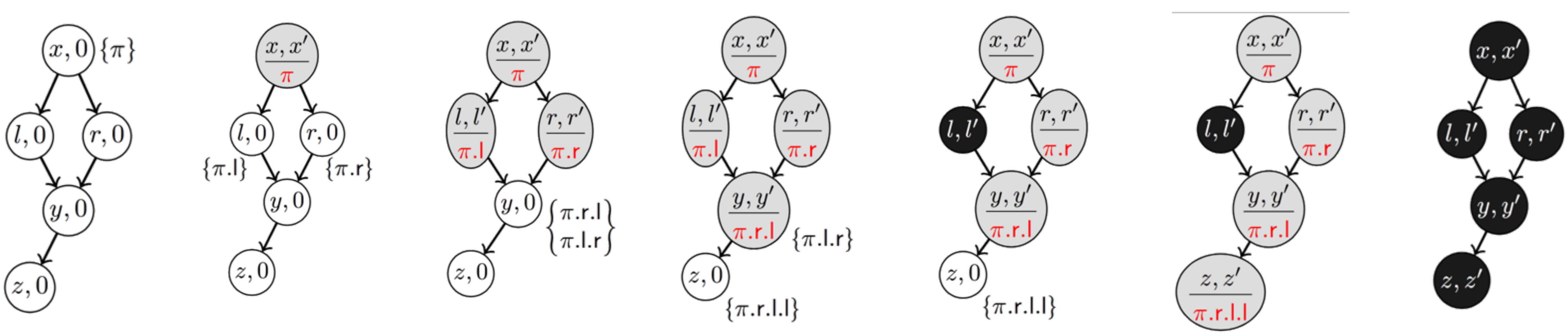
$P^{\pi}(x, \delta) \triangleq$ $\pi$ has made a promise to visit $x$; $\pi$ has made a promise to $x$ only; and $\pi$ has not spawned any threads yet:
its subthreads are not in the graph (in promise sets or as copying thread)



# 4. Mathematical Specification

Inv(δ, $\delta_c$) ≜ acyc(δ) ∧ acyc($\delta_c$) ∧ $\forall x' \in \delta_c.\ \exists! x \in \delta.$ copy($x$)= $x'$
∧ $\forall x \in \delta.\ \exists x'.$ copy($x$)=$x'$ ∧ ic($x$, $x'$, δ, $\delta_c$)

$P^{\pi}(x, \delta) \triangleq x \neq 0 \Rightarrow \pi \in$ promise($x$) ∧ $\pi$ has made a promise to $x$ only; and
$\pi$ has not spawned any threads yet:
its subthreads are not in the graph (in promise sets or as copying thread)



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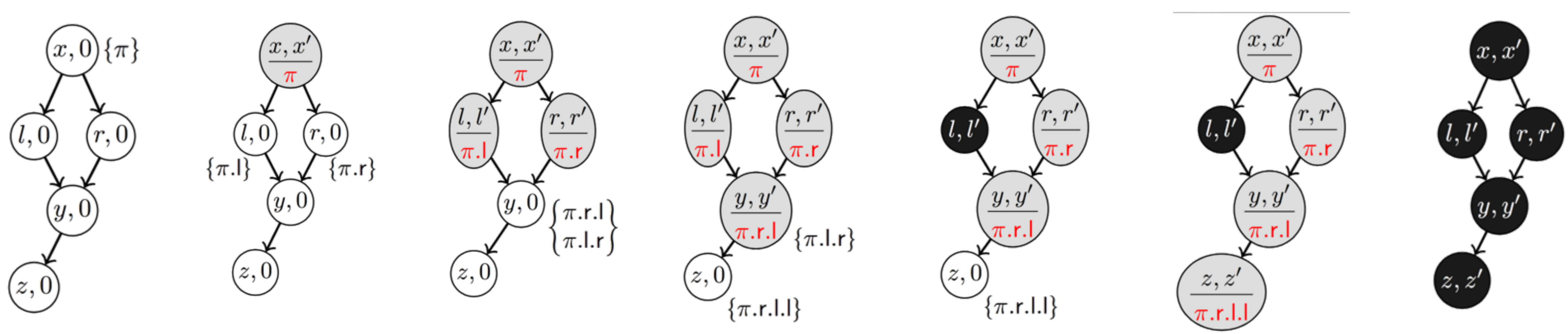
$P^{\pi}(x, \delta) \triangleq x \neq 0 \Rightarrow \pi \in \text{promise}(x) \quad \wedge \quad \forall z \in \delta.\ \pi \in \text{promise}(z) \Rightarrow x = z$

$\pi$ has not spawned any threads yet:

its subthreads are not in the graph (in promise sets or as copying thread)



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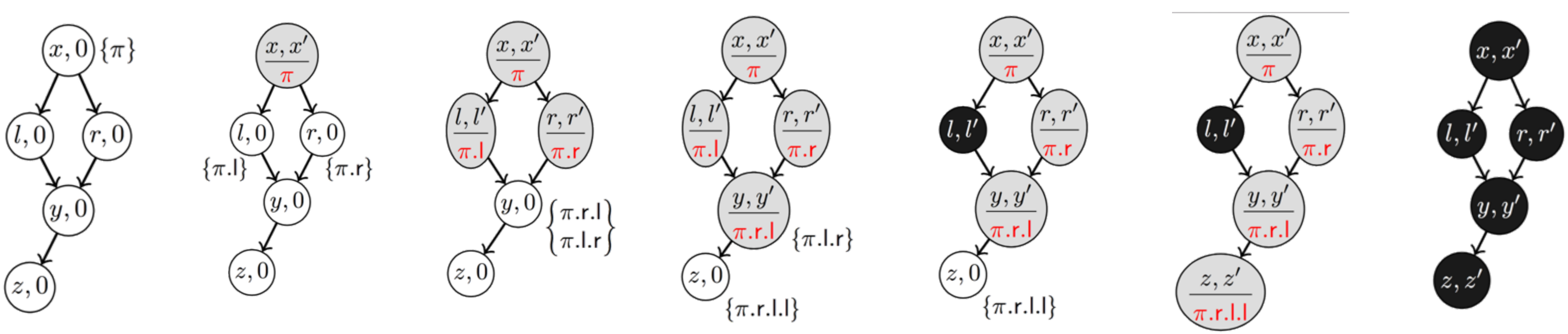
$\wedge\ \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \wedge \mathrm{ic}(x, x', \delta, \delta_c)$

$\mathrm{P}^{\pi}(x, \delta) \triangleq x \neq 0 \Rightarrow \pi \in \mathrm{promise}(x) \wedge \forall z \in \delta.\ \pi \in \mathrm{promise}(z) \Rightarrow x = z$
$\forall z \in \delta.\ \forall \pi' \sqsubset \pi.\ \pi' \notin \mathrm{promise}(z) \wedge \pi' \neq \mathrm{thread}(z)$



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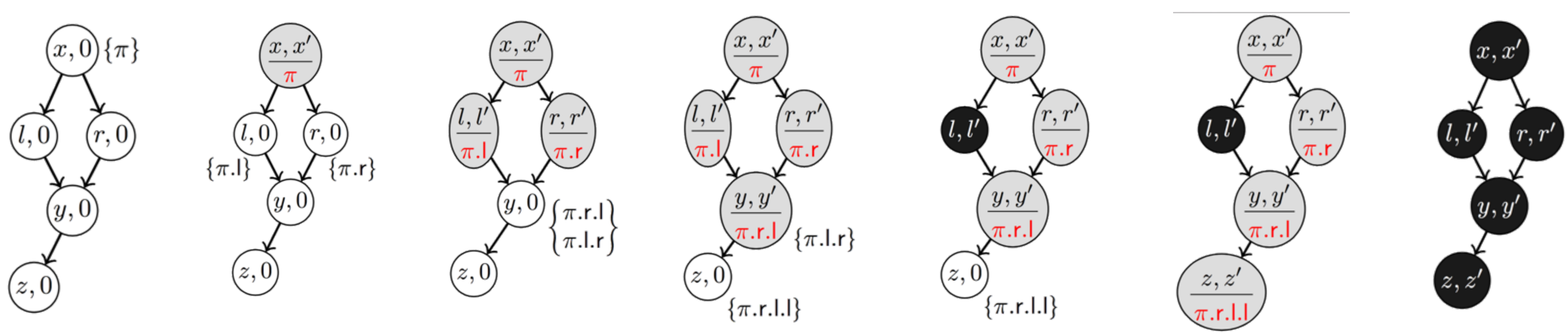
$\wedge\ \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \wedge \mathrm{ic}(x, x', \delta, \delta_c)$

$P^{\pi}(x, \delta) \triangleq x \neq 0 \Rightarrow \pi \in \mathrm{promise}(x) \wedge \forall z \in \delta.\ \pi \in \mathrm{promise}(z) \Rightarrow x = z$
$\forall z \in \delta.\ \forall \pi' \sqsubset \pi.\ \pi' \notin \mathrm{promise}(z) \wedge \pi' \neq \mathrm{thread}(z)$

$Q^{\pi}(x, x', \delta, \delta_c) \triangleq x$ is copied to $x'$ in $\delta_c$ ; and
$\pi$ and all its subthreads have finished executing (have joined):
they are not in the graph (in promise sets or as copying thread)



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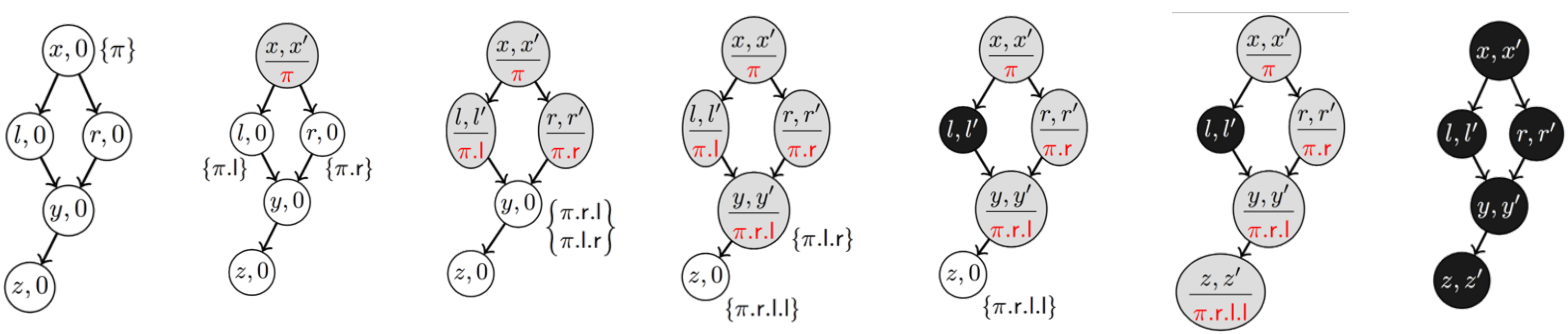
$\wedge\ \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \wedge \mathrm{ic}(x, x', \delta, \delta_c)$

$P^{\pi}(x, \delta) \triangleq x \neq 0 \Rightarrow \pi \in \mathrm{promise}(x) \wedge \forall z \in \delta.\ \pi \in \mathrm{promise}(z) \Rightarrow x = z$
$\forall z \in \delta.\ \forall \pi' \sqsubset \pi.\quad \pi' \notin \mathrm{promise}(z) \wedge \pi' \neq \mathrm{thread}(z)$

$Q^{\pi}(x, x', \delta, \delta_c) \triangleq \mathrm{copy}(x) = x' \wedge x' \in \delta_c \wedge$
$\pi$ and all its subthreads have finished executing (have joined):
they are not in the graph (in promise sets or as copying thread)



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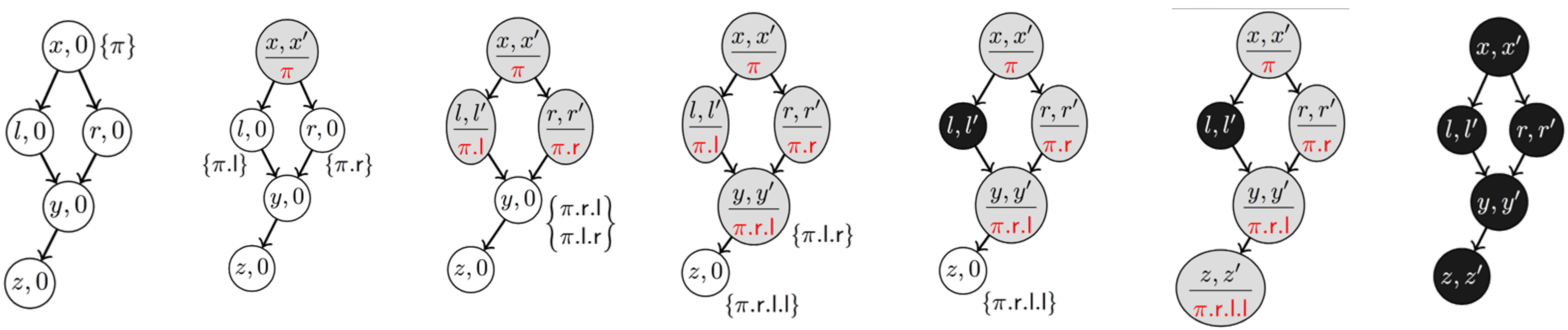

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$\forall\ \pi.\ \pi \sqsubseteq \bullet$

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$\mathrm{ic}(z, z', \delta, \delta_c)$ ≜ if $z'$ is 0 ($z$ is not copied yet), then $z$ will eventually be copied:
there exists some $y$ in δ s.t.
1) the promise set of $y$ is non-empty; 2) $y$ can reach $z$ along a path p;
and 3) every node along the path p is not copied
otherwise, $z'$ is a node in $\delta_c$ and the children of $z$ are also copied

# 4. Mathematical Specification

$\mathrm{Inv}(\delta, \delta_C) \triangleq \mathrm{acyc}(\delta) \wedge \mathrm{acyc}(\delta_C) \wedge \forall x' \in \delta_C.\ \exists! x \in \delta.\ \mathrm{copy}(x) = x'$
$\wedge\ \forall x \in \delta.\ \exists x'.\ \mathrm{copy}(x) = x' \wedge \mathrm{ic}(x, x', \delta, \delta_C)$

$Q^{\bullet}(x, x', \delta, \delta_C) \triangleq \mathrm{copy}(x) = x' \wedge x' \in \delta_C \wedge$
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$\text{ic}(z, z', \delta, \delta_c) \triangleq$ $z'$ is a node in $\delta_c$ and the children of $z$ are also copied

# 4. Mathematical Specification

Inv(δ, $δ_c$) ≜ acyc(δ) ∧ acyc($δ_c$) ∧ $\forall x' \in δ_c$. $\exists! x \in δ$. copy($x$)= $x'$
∧ $\forall x \in δ$. $\exists x'$. copy($x$)=$x'$ ∧ ic($x$, $x'$, δ, $δ_c$)

$Q^{\bullet}$($x$, $x'$, δ, $δ_c$) ≜ copy($x$) = $x'$ ∧ $x' \in δ_c$ ∧
$\forall z \in δ$. $\forall \pi'$. $\pi' \notin$ promise($z$) ∧ $\pi' \neq$ thread($x$)

ic($z$, $z'$, δ, $δ_c$) ≜ $z'$ is a node in $δ_c$ and the children of $z$ are also copied

$Q^{\bullet}$($x$, $x'$, δ, $δ_c$) ∧ Inv(δ, $δ_c$) ⇒ all nodes in δ are copied to nodes in $δ_c$

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Tracking the abstract state of the dags:
recorded in the *ghost heap; not "baked in"* to model

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$$\text{node}(x, (V, E, L)) \triangleq \exists l, r, x', P, \pi.\ E(x) = l, r \wedge L(x) = x', \pi, P \wedge$$
$$x \mapsto x', l, r \ast x \Rightarrow \pi, P$$

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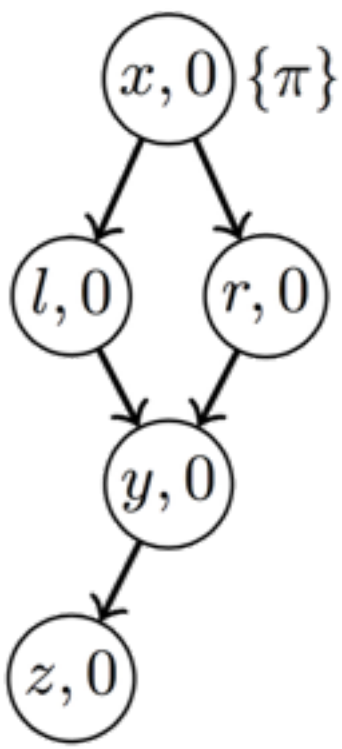
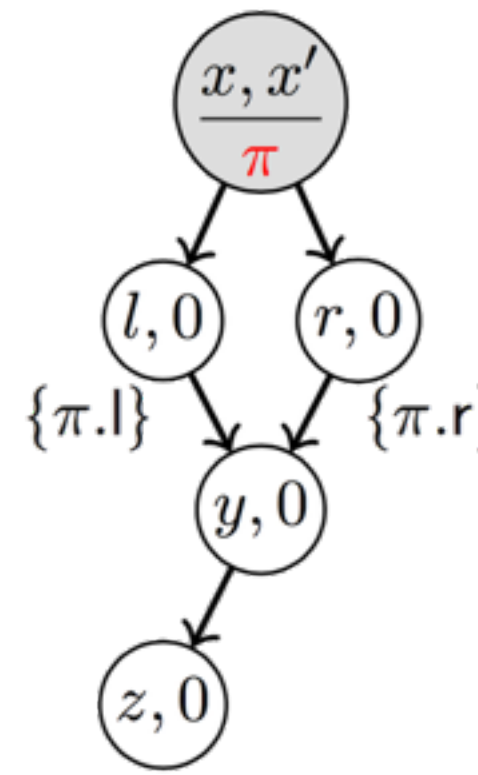



```
struct
copy_dag(
  struct
  if
  x' =
  b = <CAS(x->c, 0, x')>;
  if
    l = x->l; r = x->r;
    ll = copy_dag(l)  ||  rr = copy_dag(r)
    <x'->l = ll>; <x'->r = rr>

  } else {

  }
}
```




$(\delta, \delta_c)$ $\quad \overset{A^{\pi}}{\rightsquigarrow} \quad$ $(\delta', \delta'_c)$

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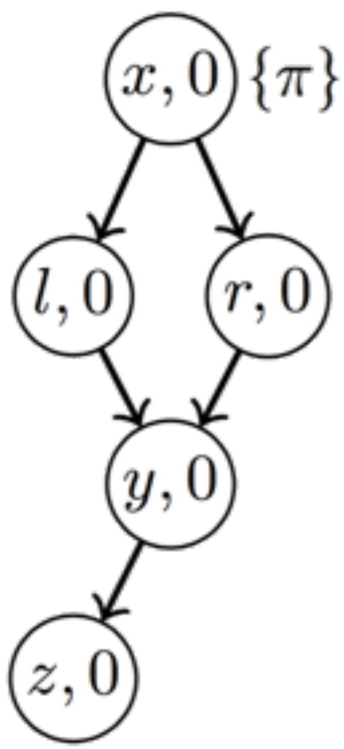


($\delta$, $\delta_c$)

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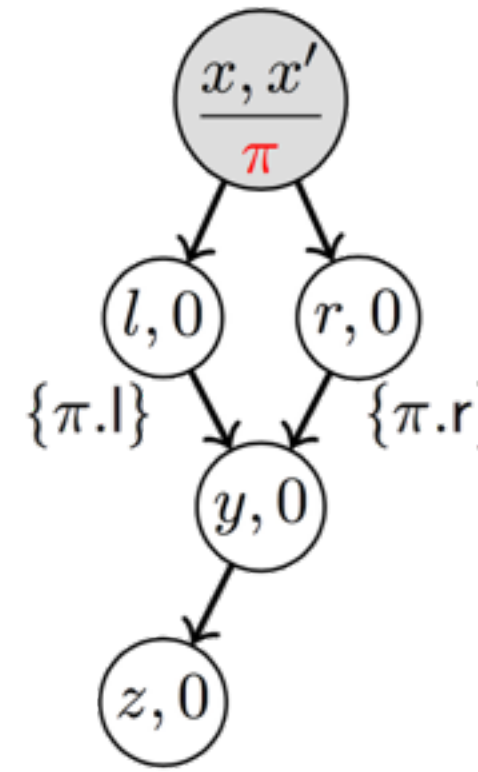
```

$\overset{A^{\pi}}{\rightsquigarrow}$

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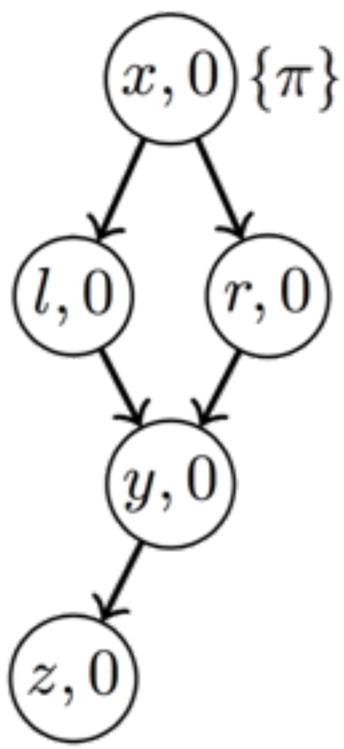


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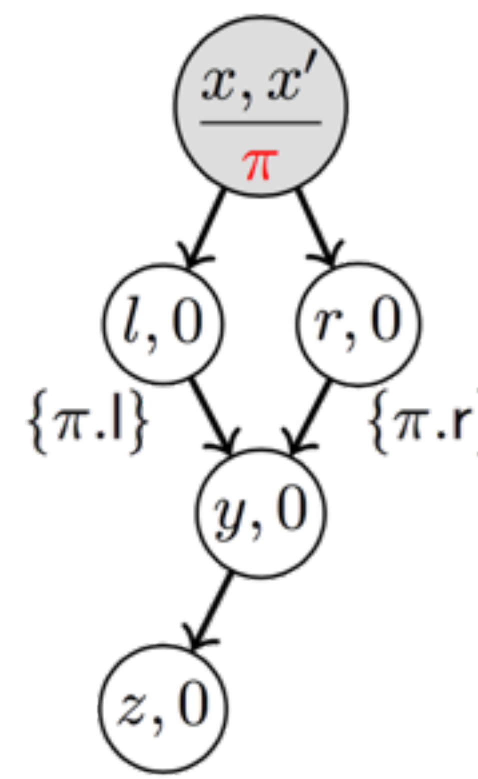


(δ, δc) $\overset{A^\pi}{\rightsquigarrow}$ (δ', δ'c)

icdag(δ, δc) $\overset{[A^\pi]}{\rightsquigarrow}$ icdag(δ', δ'c)

# 7. Spatial Specification

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- Recall
  - the mathematical invariant $\mathrm{Inv}(\delta, \delta_c)$
  - and the mathematical pre- and postconditions, $P^{\pi}(x, \delta)$ and $Q^{\pi}(x, x', \delta, \delta_c)$

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- $\delta_0$: the *original* source dag (before the top-most call to `copy_dag`)
- $\pi$: the copying thread
- $x$: the root node

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$$\mathrm{Pre}(x, \pi, \delta_0) \triangleq \underline{\pi} * \boxed{\exists \delta, \delta_c.\ \mathrm{icdag}(\delta, \delta_c) \wedge (\delta_0 \cong \delta \wedge \mathrm{Inv}(\delta, \delta_c) \wedge \mathrm{P}^{\pi}(x, \delta))}^{I}$$

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the current source dag

evolved from the original dag:
same vertices and edges
the labels may have changed

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spatial part

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pure (mathematical) part: $(\delta_0 \cong \delta \wedge \mathrm{Inv}(\delta, \delta_c) \wedge P^{\pi}(x, \delta))$

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the spatial actions allowed on dags

$$I \triangleq \{\ \pi : \mathrm{icdag}(\delta, \delta_c) \rightsquigarrow \mathrm{icdag}(\delta', \delta'_c)$$

where $\mathrm{icdag}(\delta, \delta_c)\ [A^{\pi}]\ \mathrm{icdag}(\delta', \delta'_c)$

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the local permissions for thread $\pi$ and all its descendants

$$\underline{\pi} \triangleq \mathop{\Large *}_{\pi' \in \{\pi' \mid \pi' \sqsubseteq \pi\}} \pi'$$

$$I \triangleq \{\pi : \mathrm{icdag}(\delta, \delta_c) \rightsquigarrow \mathrm{icdag}(\delta', \delta'_c)$$

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$$\text{where} \quad \mathrm{icdag}(\delta, \delta_c)\ [A^{\pi}]\ \mathrm{icdag}(\delta', \delta'_c)$$

- The spatial postcondition is

$$\mathrm{Post}(x, x', \pi, \delta_0) \triangleq \underline{\pi} * \boxed{\exists \delta, \delta_c.\ \mathrm{icdag}(\delta, \delta_c) \wedge (\delta_0 \cong \delta \wedge \mathrm{Inv}(\delta, \delta_c) \wedge Q^{\pi}(x, x', \delta, \delta_c))}_{I}$$

# Verifying `copy_dag(x)`

```
struct node {struct node *c, *l, *r};
{Pre(x, π, δ)}
copy_dag(struct node *x) {struct node *l, *r, *ll, *rr, *y; bool b;
{π* • ⟨∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^π(x, δ1))⟩_I }
  if(!x){ return 0; }
{π* • ret≐0 • ⟨∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, ret, δ1, δ2))⟩_I }
  y = malloc(sizeof(struct node));
{π* • ⟨∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^π(x, δ1))⟩_I • y ↦ 0,0,0 • y ⇒ π,∅ }
  <if(x->c){ b = false;        //Perform the action A^5_π
{π* • ⟨∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, δ1^c(x), δ1, δ2) ∧ δ1^c(x) ≠ 0)⟩_I
 • y ↦ 0,−,− • y ⇒ π,∅ • b≐0 }
  }else{ x->c = y; b = true;   //Perform the action A^1_π
{π* • ⟨∃δ1,δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
   (x≠y ⇒ π≠δ1^s(y)) ∧ ∃l,r. δ1(x)=(y, π, −, l, r) ∧ y∉δ2 ∧ P^π.l(l, δ1) ∧ P^π.r(r, δ1))⟩_I • b≐1 }
  }>
  if(b){ l = x->l; r = x->r;
{π* • ⟨∃δ1,δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
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   (x≠y ⇒ π≠δ1^s(y)) ∧ δ1(x)=(y,−,π,l,r) ∧ y∉δ2)⟩_I
 • π.l* • ⟨∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^π.l(l, δ1))⟩_I
 • π.r* • ⟨∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^π.r(r, δ1))⟩_I }
{π • ⟨∃δ1,δ2. icdag(δ1,δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y)
   ∧ (x≠y ⇒ π≠δ1^s(y)) ∧ δ1(x)=(y,−,π,l,r) ∧ y∉δ2)⟩_I • Pre(l, π.l, δ) • Pre(r, π.r, δ) }

        {Pre(l, π.l, δ)}          ||       {Pre(r, π.r, δ)}
        ll = copy_dag(l)          ||       rr = copy_dag(r)
        {Post(l, ll, π.l, δ)}     ||       {Post(r, rr, π.r, δ)}

{π • ⟨∃δ1,δ2. icdag(δ1,δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y)
   ∧ (x≠y ⇒ π≠δ1^s(y)) ∧ δ1(x)=(y,−,π,l,r) ∧ y∉δ2)⟩_I • Post(l, ll, π.l, δ) • Post(r, rr, π.r, δ) }
{π* • ⟨∃δ1,δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1,δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
   (x≠y ⇒ π≠δ1^s(y)) ∧ δ1(x)=(y,−,π,l,r) ∧ y∉δ2 ∧ Q^π.l(l, ll, δ1, δ2) ∧ Q^π.r(r, rr, δ1, δ2))⟩_I }
   <y->l = ll>; <y->r = rr>;  //Perform A^2_π, A^3_π and A^4_π in order.
{π* • ⟨∃δ1,δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, y, δ1, δ2))⟩_I }
   return y;  {π* • ⟨∃δ1,δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, ret, δ1, δ2))⟩_I }
  }else{
{π* • ⟨∃δ1,δ2. icdag(δ1,δ2) • (δ≐δ1 ∧ Inv(δ1,δ2) ∧ Q^π(x, δ1^c(x), δ1, δ2) ∧ δ1^c(x) ≠ 0)⟩_I • y ↦ 0,−,− • y ⇒ π,∅ }
   free(y, sizeof(struct node)) ; return x->c;
{π* • ⟨∃δ1,δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, ret, δ1, δ2))⟩_I }
} }  {Post(x, ret, π, δ)}
```

Changes reflected in the pure (mathematical) part as highlighted

# Verifying `copy_dag(x)`

```
struct node {struct node *c, *l, *r};
{Pre(x, π, δ)}
copy_dag(struct node *x) {struct node *l, *r, *ll, *rr, *y; bool b;
{π* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^π(x, δ1)))_I }
  if(!x){ return 0; }
{π* • ret≐0 • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, ret, δ1, δ2)))_I }
  y = malloc(sizeof(struct node));
{π* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^π(x, δ1)))_I • y↦0,0,0 • y⇒π,∅ }
  <if(x->c){ b = false;      //Perform the action A^5_π
{π* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, δ1^c(x), δ1, δ2) ∧ δ1^c(x) ≠ 0))_I
 • y↦0,−,− • y⇒π,∅ • b≐0 }
  }else{ x->c = y; b = true;      //Perform the action A^1_π
{π* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
  (x≠y ⇒ π≠δ1^c(y)) ∧ ∃l, r. δ1(x)=(y, π, −, l, r) ∧ y∈̇δ2 ∧ P^{π.l}(l, δ1) ∧ P^{π.r}(r, δ1)))_I • b≐1 }
  }>
  if(b){ l = x->l; r = x->r;
{π* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
  (x≠y ⇒ π≠δ1^c(y)) ∧ δ1(x)=(y, π, −, l, r) ∧ y∈̇δ2 ∧ P^{π.l}(l, δ1) ∧ P^{π.r}(r, δ1)))_I }
{π • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
  (x≠y ⇒ π≠δ1^c(y)) ∧ δ1(x)=(y, −, π, l, r) ∧ y∈̇δ2))_I
 • π.l* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^{π.l}(l, δ1)))_I
 • π.r* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ P^{π.r}(r, δ1)))_I }
{π • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y)
  ∧ (x≠y ⇒ π≠δ1^c(y)) ∧ δ1(x)=(y, −, π, l, r) ∧ y∈̇δ2))_I • Pre(l, π.l, δ) • Pre(r, π.r, δ) }
        {Pre(l, π.l, δ)}        ||   {Pre(r, π.r, δ)}
        ll = copy_dag(l)        ||   rr = copy_dag(r)
        {Post(l, ll, π.l, δ)}   ||   {Post(r, rr, π.r, δ)}
{π • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y)
  ∧ (x≠y ⇒ π≠δ1^c(y)) ∧ δ1(x)=(y, −, π, l, r) ∧ y∈̇δ2))_I • Post(l, ll, π.l, δ) • Post(r, rr, π.r, δ) }
{π* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ ∀y∈δ1. π∉δ1^p(y) ∧
  (x≠y ⇒ π≠δ1^c(y)) ∧ δ1(x)=(y, −, π, l, r) ∧ y∈̇δ2 ∧ Q^{π.l}(l, ll, δ1, δ2) ∧ Q^{π.r}(r, rr, δ1, δ2)))_I }
   <y->l = ll>; <y->r = rr>;  //Perform A^2_π, A^3_π and A^4_π in order.
{π* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, y, δ1, δ2)))_I }
   return y; {π* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, ret, δ1, δ2)))_I }
  }else{
{π* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, δ1^c(x), δ1, δ2) ∧ δ1^c(x) ≠ 0))_I • y↦0,−,− • y⇒π,∅ }
   free(y, sizeof(struct node)) ; return x->c;
{π* • (∃δ1, δ2. icdag(δ1, δ2) • (δ≐δ1 ∧ Inv(δ1, δ2) ∧ Q^π(x, ret, δ1, δ2)))_I }
} } {Post(x, ret, π, δ)}
```

Changes reflected in the pure (mathematical) part as highlighted

The spatial part *appears* unchanged as highlighted

# Verifying `copy_dag(x)`


Changes reflected in the pure (mathematical) part as highlighted

The spatial part *appears* unchanged as highlighted

# Conclusions

- ☑ Verified 4 concurrent fine-grained graph algorithms
  - ☑ Copying dags (directed acyclic graphs)
  - ☑ Speculative variant of Dijkstra's shortest path
  - ☑ Computing the spanning tree of a graph
  - ☑ Marking a graph
- ☑ Presented a common proof pattern for graph algorithms
  - ☑ *Abstract mathematical* graphs for Functional correctness
  - ☑ *Concrete Spatial (heap-represented)* graphs for memory safety
  - ☑ Combined reasoning for full proof
  - ☑ Inspired by existing logics where this pattern is "baked-in" to the model
  - ☑ "Baking-in" is unnecessary; demonstrated by CoLoSL reasoning

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**Thank you for listening!**

# Speculative Concurrent Shortest Path

```
parallel_dijkstra((int[][] a, int[] c, int size, src) {
  bitarray work[size], done[size];
  for (i=0; i<size; i++){
    c[i] = a[src][i]; work[i] = 1; done[i] = 0;  //initialisation
  }; c[src] = 0;
  dijkstra(a,c,size,work,done) || ... || dijkstra(a,c,size,work,done)
}

dijkstra(int[][] a, int[] c, int size, bitarray work, done){
  i = 0;
  while(done != 2^size-1){
    b = <CAS(work[i], 1, 0)>;
    if(b){ cost = c[i];
      for(j=0; j<size; j++){ newcost = cost + a[i][j]; b = true;
        do{ oldcost = c[j];
          if(newcost < oldcost){
            b = <CAS(work[j], 1, 0)>;
            if(b){ b = <CAS(c[j], oldcost, newcost)>; <work[j] = 1>; }
            else { b = <CAS(done[j], 1, 0)>;
              if(b){ b = <CAS(c[j], oldcost, newcost)>;
                if(b){ < work[j] = 1 > } else { < done[j] = 1 > }
          } } }
        } while(!b)
      } < done[i] = 1 >;
    } i = (i+1) mod size;
} }
```