A PROOF OF THEOREM 1

Proof. Pick an arbitrary execution $G = (E, \mathsf{po}, \mathsf{rf}, \mathsf{mo})$ such that $E \subseteq R \cup W \cup U \cup MF$.

RTS: if $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{uc}}$ and G is Ex86-consistent, then G is SC-consistent.
Let us assume $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{uc}}$ and G is Ex86-consistent. From COL, ROW and ROW2 in Def. 5 we then have $\mathsf{ppo} = \mathsf{po}$ and thus from the definition of $\mathsf{ppo}_{\mathrm{SC}}$ we have $\mathsf{ppo}_{\mathrm{SC}} = \mathsf{ppo}$. On the other hand, since G is Ex86-consistent, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. Consequently, since $\mathsf{ppo}_{\mathrm{SC}} = \mathsf{ppo}$, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo}_{\mathrm{SC}} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. As such, from the definition of SC-consistency we have G is SC-consistent, as required.

RTS: if $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{uc}}$ and G is SC-consistent, then G is Ex86-consistent.
Let us assume $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{uc}}$ and G is SC-consistent. From Def. 7 we then have $\mathsf{ppo}_{\mathrm{SC}} = \mathsf{po}$. Similarly, from COL, ROW and ROW2 in the definition of ppo (Def. 5) we then have $\mathsf{ppo} = \mathsf{po}$ and thus from the definition of $\mathsf{ppo}_{\mathrm{SC}}$ we have $\mathsf{ppo}_{\mathrm{SC}} = \mathsf{ppo}$. On the other hand, since G is SC-consistent, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo}_{\mathrm{SC}} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. Consequently, since $\mathsf{ppo}_{\mathrm{SC}} = \mathsf{ppo}$, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. As such, from the definition of Ex86-consistency we have G is Ex86-consistent, as required.

RTS: if $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{c}}$ and G is Ex86-consistent, then G is TSO-consistent.
Let us assume $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{c}}$ and G is Ex86-consistent. From COL, ROW and W-WB in Def. 5 we then have $\mathsf{ppo} = \mathsf{po} \setminus (W \times R)$ and thus from the definition of $\mathsf{ppo}_{\mathrm{TSO}}$ we have $\mathsf{ppo}_{\mathrm{TSO}} = \mathsf{ppo}$. On the other hand, since G is Ex86-consistent, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. Consequently, since $\mathsf{ppo}_{\mathrm{TSO}} = \mathsf{ppo}$, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo}_{\mathrm{TSO}} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. As such, from the definition of TSO-consistency we have G is TSO-consistent, as required.

RTS: if $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{c}}$ and G is TSO-consistent, then G is Ex86-consistent.
Let us assume $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{c}}$ and G is TSO-consistent. From Def. 7 we then have $\mathsf{ppo}_{\mathrm{TSO}} = \mathsf{po} \setminus (W \times R)$. Similarly, from COL, ROW and ROW2 in the definition of ppo (Def. 5) we then have $\mathsf{ppo} = \mathsf{po} \setminus (W \times R)$ and thus from the definition of $\mathsf{ppo}_{\mathrm{TSO}}$ we have $\mathsf{ppo}_{\mathrm{TSO}} = \mathsf{ppo}$. On the other hand, since G is TSO-consistent, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo}_{\mathrm{TSO}} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. Consequently, since $\mathsf{ppo}_{\mathrm{TSO}} = \mathsf{ppo}$, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. As such, from the definition of Ex86-consistency we have G is Ex86-consistent, as required.

RTS: if $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{wc}}$, then G is SPSO-consistent.
Let us assume $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{wc}}$ and G is Ex86-consistent. From COL, ROW and W-LOC in Def. 5 we have $\mathsf{ppo} = \mathsf{po} \setminus ((W \times W) \setminus \mathsf{sloc})$ and thus from the definition of $\mathsf{ppo}_{\mathrm{SPSO}}$ we have $\mathsf{ppo}_{\mathrm{SPSO}} = \mathsf{ppo}$. On the other hand, since G is Ex86-consistent, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. Consequently, since $\mathsf{ppo}_{\mathrm{SPSO}} = \mathsf{ppo}$, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo}_{\mathrm{SPSO}} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. As such, from the definition of SPSO-consistency we have G is SPSO-consistent, as required.

RTS: if $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{wc}}$ and G is SPSO-consistent, then G is Ex86-consistent.
Let us assume $\mathtt{loc}(E) \subseteq \mathrm{Loc}_{\mathrm{wc}}$ and G is SPSO-consistent. From Def. 7 we then have $\mathsf{ppo}_{\mathrm{SPSO}} = \mathsf{po} \setminus ((W \times W) \setminus \mathsf{sloc})$. Similarly, from COL, ROW and W-LOC in Def. 5 we have $\mathsf{ppo} = \mathsf{po} \setminus ((W \times W) \setminus \mathsf{sloc})$ and thus from the definition of $\mathsf{ppo}_{\mathrm{SPSO}}$ we have $\mathsf{ppo}_{\mathrm{SPSO}} = \mathsf{ppo}$. On the other hand, since G is SPSO-consistent, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo}_{\mathrm{SPSO}} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. Consequently, since $\mathsf{ppo}_{\mathrm{SPSO}} = \mathsf{ppo}$, we have $\mathsf{rf}_i \cup \mathsf{mo}_i \cup \mathsf{rb}_i \subseteq \mathsf{po}$ and $\mathrm{acyclic}(\mathsf{ppo} \cup \mathsf{rf}_e \cup \mathsf{mo}_e \cup \mathsf{rb}_e)$. As such, from the definition of Ex86-consistency we have G is Ex86-consistent, as required. ☐

B THE EVENT-ANNOTATED, OPERATIONAL PEx86 SEMANTICS

Annotated persistent memory

$$M \in \textsc{AMem} \triangleq \left\{ f \in \textsc{Loc} \xrightarrow{\text{fin}} ST \;\middle|\; \forall x \in dom(f).\ \texttt{loc}(f(x)) = x \right\}$$

Annotated persistent buffers

$$PB \in \textsc{APBMap} \triangleq \left\{ f \in \textsc{Loc}_{\text{wb}} \mapsto \textsc{APBuff} \;\middle|\; \begin{array}{l} \forall x \in dom(f), e \in f(x). \\ \quad e \in W \cup U \Rightarrow \texttt{loc}(e) = x \\ \quad \land\ e \in FO \Rightarrow (x, \texttt{loc}(e)) \in \mathsf{scl} \end{array} \right\}$$

$$pb \in \textsc{APBuff} \triangleq \textsc{Seq}\,\langle \textsc{PBEvent} \rangle \quad \text{with} \quad \textsc{PBEvent} \triangleq W_{\text{wb}} \cup U_{\text{wb}} \cup FO$$

Annotated volatile buffers

$$b \in \textsc{ABuff} \triangleq \textsc{Seq}\,\langle \textsc{BEvent} \rangle \quad \text{with} \quad \textsc{BEvent} \triangleq W \cup NTW \cup FL \cup FO \cup SF$$

$$B \in \textsc{ABMap} \triangleq \left\{ f \in \textsc{TId} \xrightarrow{\text{fin}} \textsc{ABuff} \;\middle|\; \forall \tau.\ \forall e \in f(\tau).\ \texttt{tid}(e) = \tau \right\}$$

Annotated labels

$$\begin{array}{lll}
\textsc{ALabels} \ni \lambda ::= & \mathsf{R}\langle r, e \rangle & \text{where } r \in R, e \in ST, \texttt{loc}(r){=}\texttt{loc}(e), \texttt{val}_{\mathsf{r}}(r){=}\texttt{val}_{\mathsf{w}}(e) \\
& \mid \mathsf{U}\langle u, e \rangle & \text{where } u \in U, e \in ST, \texttt{loc}(u){=}\texttt{loc}(e), \texttt{val}_{\mathsf{r}}(u){=}\texttt{val}_{\mathsf{w}}(e) \\
& \mid \mathsf{W}\langle w \rangle & \text{where } w \in W \\
& \mid \mathsf{NTW}\langle w \rangle & \text{where } w \in NTW \\
& \mid \mathsf{MF}\langle mf \rangle & \text{where } mf \in MF \\
& \mid \mathsf{SF}\langle sf \rangle & \text{where } sf \in SF \\
& \mid \mathsf{FO}\langle fo \rangle & \text{where } fo \in FO \\
& \mid \mathsf{FL}\langle fl \rangle & \text{where } fl \in FL \\
& \mid \mathsf{B}\langle e \rangle & \text{where } e \in SF \cup W_{\text{wb}} \\
& \mid \mathsf{B}\langle fo, S \rangle & \text{where } fo \in FO \land S \subseteq ST_{\text{wb}} \land \mathsf{sameCL}(\texttt{loc}(fo), S) \\
& \mid \mathsf{P}\langle fl, S \rangle & \text{where } fl \in FL \land S \subseteq ST_{\text{wb}} \land \mathsf{sameCL}(\texttt{loc}(fl), S) \\
& \mid \mathsf{P}\langle fo, w \rangle & \text{where } e \in FO \land w \in ST \land (\texttt{loc}(e), \texttt{loc}(w)) \in \mathsf{scl}_{\text{wb}} \\
& \mid \mathsf{P}\langle e \rangle & \text{where } e \in W \cup NTW \cup U_{\text{wb}} \\
& \mid \mathcal{E}\langle \tau \rangle & \text{where } \tau \in \textsc{TId}
\end{array}$$

$$\mathsf{sameCL}(x, S) \overset{\text{def}}{\Leftrightarrow} \forall w \in S.\ (x, \texttt{loc}(w)) \in \mathsf{scl} \land \forall y.\ (x, y) \in \mathsf{scl} \Rightarrow \exists! w \in S.\ w \in ST_y$$

$$\pi \in \textsc{Path} \triangleq \textsc{Seq}\left\langle \textsc{ALabels} \setminus \left\{ \mathcal{E}\langle \tau \rangle \;\middle|\; \tau \in \textsc{TId} \right\} \right\rangle \qquad \text{Event paths}$$

B.1 Storage Subsystem

Let:

$$\texttt{rd}(M, pb, b, x) \triangleq \begin{cases} e & \text{if } \exists b_1, b_2.\ b = b_1.e.b_2 \land e \in ST_x \land b_2 \cap ST_x = \emptyset \\ e & \text{else if } \exists pb_1, pb_2.\ pb = pb_1.e.pb_2 \land e \in ST_x \land pb_2 \cap ST_x = \emptyset \\ M(x) & \text{otherwise} \end{cases}$$

and

$$\mathsf{PO}(b) \triangleq \left\{ (e_1, e_2) \;\middle|\; \exists n_1, n_2.\ b\#_{n_1}{=}e_1 \land b\#_{n_2}{=}e_2 \land n_1 < n_2 \right\}$$
$$\mathsf{PPO}(b) \triangleq \mathsf{ppo}(\mathsf{PO}(b))$$

Given a set of events E, let us write E_τ for $\{e \in E \mid \mathtt{tid}(e) = \tau\}$. The annotated transitions are then given as follows:

$$\frac{\mathtt{tid}(r) = \tau \qquad \mathtt{loc}(r) = x \qquad x \in \textsc{Loc}_{\mathsf{c}} \qquad \mathtt{rd}(M, PB(x), B(\tau), x) = e}{M, PB, B \xrightarrow{\mathsf{R}\langle r,e \rangle} M, PB, B} \ \textsc{AM-ReadC}$$

$$\frac{\mathtt{tid}(r) = \tau \qquad \mathtt{loc}(r) = x \qquad x \in \textsc{Loc}_{\mathsf{nc}} \qquad B(\tau) = \epsilon \qquad M(x) = e}{M, PB, B \xrightarrow{\mathsf{R}\langle r,e \rangle} M, PB, B} \ \textsc{AM-ReadNC}$$

$$\frac{\mathtt{tid}(w) = \tau \qquad B(\tau) = b \qquad b' = b.w}{M, PB, B \xrightarrow{\mathsf{W}\langle w \rangle} M, PB, B[\tau \mapsto b']} \ \textsc{AM-Write}$$

$$\frac{\mathtt{tid}(w) = \tau \qquad B(\tau) = b \qquad b' = b.w}{M, PB, B \xrightarrow{\mathsf{NTW}\langle w \rangle} M, PB, B[\tau \mapsto b']} \ \textsc{AM-NTWrite}$$

$$\frac{\mathtt{tid}(fl) = \tau \qquad B(\tau) = b \qquad b' = b.fl}{M, PB, B \xrightarrow{\mathsf{FL}\langle fl \rangle} M, PB, B[\tau \mapsto b']} \ \textsc{AM-FL}$$

$$\frac{\mathtt{tid}(fo) = \tau \qquad B(\tau) = b \qquad b' = b.fo}{M, PB, B \xrightarrow{\mathsf{FO}\langle fo \rangle} M, PB, B[\tau \mapsto b']} \ \textsc{AM-FO}$$

$$\frac{\mathtt{tid}(sf) = \tau \qquad B(\tau) = b \qquad b' = b.sf}{M, PB, B \xrightarrow{\mathsf{SF}\langle sf \rangle} M, PB, B[\tau \mapsto b']} \ \textsc{AM-SF}$$

$$\frac{\mathtt{tid}(mf) = \tau \qquad B(\tau) = \epsilon \qquad \forall y.\ PB(y) \cap FO_\tau = \emptyset}{M, PB, B \xrightarrow{\mathsf{MF}\langle mf \rangle} M, PB, B} \ \textsc{AM-MF}$$

$$\frac{\begin{array}{c}\mathtt{tid}(u) = \tau \qquad \mathtt{loc}(u) = x \qquad B(\tau) = \epsilon \qquad \forall y.\ PB(y) \cap FO_\tau = \emptyset \\ x \in \textsc{Loc}_{\mathsf{wb}} \qquad \mathtt{rd}(M, PB(x), \epsilon, x) = e \qquad PB' = PB[x \mapsto PB(x).u]\end{array}}{M, PB, B \xrightarrow{\mathsf{U}\langle u,e \rangle} M, PB', B} \ \textsc{AM-RMW1}$$

$$\frac{\begin{array}{c}\mathtt{tid}(u) = \tau \qquad \mathtt{loc}(u) = x \qquad B(\tau) = \epsilon \qquad \forall y.\ PB(y) \cap FO_\tau = \emptyset \\ x \notin \textsc{Loc}_{\mathsf{wb}} \qquad M(x) = e \qquad M' = M[x \mapsto u]\end{array}}{M, PB, B \xrightarrow{\mathsf{U}\langle u,e \rangle} M', PB, B} \ \textsc{AM-RMW2}$$

$$\frac{\mathtt{tid}(sf) = \tau \qquad B(\tau) = sf.b \qquad sf \in SF \qquad \forall y.\ PB(y) \cap FO_\tau = \emptyset}{M, PB, B \xrightarrow{\mathsf{B}\langle sf \rangle} M, PB, B[\tau \mapsto b]} \ \textsc{AM-PropSF}$$

$$\frac{B(\tau)=b_1.w.b_2 \quad w \in W \quad \texttt{loc}(w)=x \quad x \in \text{Loc}_{\text{wb}} \quad PB(x)=pb \qquad \forall e \in b_1.\ (e,w) \notin \text{PPO}(B(\tau))}{M, PB, B \xrightarrow{\text{B}\langle w \rangle} M, PB[x \mapsto pb.w], B[\tau \mapsto b_1.b_2]} \text{ AM-PropW1}$$

$$\frac{B(\tau)=b_1.w.b_2 \quad w \in W \quad \texttt{loc}(w)=x \quad x \notin \text{Loc}_{\text{wb}} \qquad \forall e \in b_1.\ (e,w) \notin \text{PPO}(B(\tau))}{M, PB, B \xrightarrow{\text{P}\langle w \rangle} M[x \mapsto w], PB, B[\tau \mapsto b_1.b_2]} \text{ AM-PropW2}$$

$$\frac{B(\tau)=b_1.w.b_2 \quad w \in NTW \quad \texttt{loc}(w)=x \quad x \in \text{Loc}_{\text{wb}} \Rightarrow PB(x)=\epsilon \qquad \forall e \in b_1.\ (e,w) \notin \text{PPO}(B(\tau))}{M, PB, B \xrightarrow{\text{P}\langle w \rangle} M[x \mapsto w], PB, B[\tau \mapsto b_1.b_2]} \text{ AM-PropNTW}$$

$$\frac{B(\tau)=b_1.fl.b_2 \quad fl \in FL \quad \texttt{loc}(fl)=x \quad \forall y.\ (x,y) \in \text{scl} \Rightarrow PB(y)=\epsilon \qquad \forall e \in b_1.\ (e,fl) \notin \text{PPO}(B(\tau)) \quad S=\left\{M(y) \,\middle|\, (x,y) \in \text{scl}\right\}}{M, PB, B \xrightarrow{\text{P}\langle fl, S \rangle} M, PB, B[\tau \mapsto b_1.b_2]} \text{ AM-PropFL}$$

$$\frac{B(\tau)=b_1.fo.b_2 \quad fo \in FO \quad \texttt{loc}(fo)=x \quad \forall e \in b_1.\ (e,fo) \notin \text{PPO}(B(\tau)) \qquad PB'=\lambda y.\ (y,x) \in \text{scl}\ ?\ PB(y).fo : \text{PB}(y) \quad S=\left\{\texttt{rd}(M, PB(y), \epsilon, y) \,\middle|\, (x,y) \in \text{scl}\right\}}{M, PB, B \xrightarrow{\text{B}\langle fo, S \rangle} M, PB', B[\tau \mapsto b_1.b_2]} \text{ AM-PropFO}$$

$$\frac{\texttt{loc}(w)=x \quad PB(x)=w.pb \quad w \in W \cup U}{M, PB, B \xrightarrow{\text{P}\langle w \rangle} M[x \mapsto w], PB[x \mapsto pb], B} \text{ AM-PersistW}$$

$$\frac{\texttt{loc}(fo)=x \quad PB(x)=fo.pb \quad fo \in FO \quad \text{M}(x)=w}{M, PB, B \xrightarrow{\text{P}\langle fo, w \rangle} M, PB[x \mapsto pb], B} \text{ AM-PersistFO}$$

Thread Subsystem

Thread-local steps.

$$\frac{\text{C}_1, \xrightarrow{\lambda} \text{C}_1'}{\textbf{let } a{:=}\text{C}_1 \textbf{ in } \text{C}_2 \xrightarrow{\lambda} \textbf{let } a{:=}\text{C}_1' \textbf{ in } \text{C}_2} \text{ AT-Let1} \qquad \frac{}{\textbf{let } a{:=}v \textbf{ in } \text{C} \xrightarrow{\mathcal{E}\langle \tau \rangle} \text{C}[v/a]} \text{ AT-Let2}$$

$$\frac{\text{C}, \xrightarrow{\lambda} \text{C}'}{\textbf{if } (\text{C}) \textbf{ then } \text{C}_1 \textbf{ else } \text{C}_2 \xrightarrow{\lambda} \textbf{if } (\text{C}') \textbf{ then } \text{C}_1 \textbf{ else } \text{C}_2} \text{ AT-If1}$$

$$\frac{v{\neq}0 \Rightarrow \text{C}{=}\text{C}_1 \quad v{=}0 \Rightarrow \text{C}{=}\text{C}_2}{\textbf{if } (v) \textbf{ then } \text{C}_1 \textbf{ else } \text{C}_2 \xrightarrow{\mathcal{E}\langle \tau \rangle} \text{C}} \text{ T-If2}$$

$$\frac{}{\textbf{repeat}\ \mathsf{C} \xrightarrow{\mathcal{E}\langle\tau\rangle} \textbf{if}\ (\mathsf{C})\ \textbf{then}\ (\textbf{repeat}\ \mathsf{C})\ \textbf{else}\ 0}\ \textsc{T-Repeat}$$

$$\frac{\mathtt{val_w}(w){=}v \quad \mathtt{loc}(w){=}x}{\textbf{store}(x,v) \xrightarrow{\mathsf{W}\langle w\rangle} v}\ \textsc{AT-Write} \qquad \frac{\mathtt{val_w}(w){=}v \quad \mathtt{loc}(w){=}x}{\textbf{ntstore}(x,v) \xrightarrow{\mathsf{NTW}\langle w\rangle} v}\ \textsc{AT-NTW}$$

$$\frac{\mathtt{val_r}(r){=}v \quad \mathtt{loc}(r){=}x}{\textbf{load}(x) \xrightarrow{\mathsf{R}\langle r,w\rangle} v}\ \textsc{AT-Read} \qquad \frac{}{\textbf{mfence} \xrightarrow{\mathsf{MF}\langle mf\rangle} 1}\ \textsc{AT-MFence}$$

$$\frac{\mathtt{val_r}(r) \neq v_1 \quad \mathtt{loc}(r){=}x}{\textbf{CAS}(x,v_1,v_2) \xrightarrow{\mathsf{R}\langle r,w\rangle} 0}\ \textsc{AT-CAS0} \qquad \frac{\mathtt{val_r}(u){=}v_1 \quad \mathtt{val_w}(u){=}v_2 \quad \mathtt{loc}(u){=}x}{\textbf{CAS}(x,v_1,v_2) \xrightarrow{\mathsf{U}\langle u,w\rangle} 1}\ \textsc{AT-CAS1}$$

$$\frac{}{\textbf{sfence} \xrightarrow{\mathsf{SF}\langle sf\rangle} 1}\ \textsc{AT-SFence} \qquad \frac{\mathtt{loc}(fo){=}x}{\textbf{flush}_{\text{opt}}\ x \xrightarrow{\mathsf{FO}\langle fo\rangle} 1}\ \textsc{AT-FO} \qquad \frac{\mathtt{loc}(fl){=}x}{\textbf{flush}\ x \xrightarrow{\mathsf{FL}\langle fl\rangle} 1}\ \textsc{AT-FL}$$

Program Steps.

$$\frac{\mathsf{P}(\tau) \xrightarrow{\lambda} \mathsf{C} \quad \mathtt{tid}(\lambda) = \tau}{\mathsf{P} \xrightarrow{\lambda} \mathsf{P}[\tau \mapsto \mathsf{C}]}\ \textsc{AP-Step}$$

where:

$$\mathtt{tid}(\lambda) \triangleq \begin{cases} \tau & \text{if } \lambda{=}\mathcal{E}\langle\tau\rangle \\ \mathtt{tid}(\mathtt{event}(\lambda)) & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\mathtt{event}(\mathsf{R}\langle r,w\rangle) &\triangleq r \\
\mathtt{event}(\mathsf{U}\langle u,w\rangle) &\triangleq u \\
\mathtt{event}(\mathsf{W}\langle w\rangle) &\triangleq w \\
\mathtt{event}(\mathsf{NTW}\langle w\rangle) &\triangleq w \\
\mathtt{event}(\mathsf{MF}\langle mf\rangle) &\triangleq mf \\
\mathtt{event}(\mathsf{SF}\langle sf\rangle) &\triangleq sf \\
\mathtt{event}(\mathsf{FO}\langle fo\rangle) &\triangleq fo \\
\mathtt{event}(\mathsf{FL}\langle fl\rangle) &\triangleq fl \\
\mathtt{event}(\mathsf{B}\langle e\rangle) &\triangleq e \\
\mathtt{event}(\mathsf{B}\langle e,S\rangle) &\triangleq e \\
\mathtt{event}(\mathsf{P}\langle e,S\rangle) &\triangleq e \\
\mathtt{event}(\mathsf{P}\langle e,w\rangle) &\triangleq e \\
\mathtt{event}(\mathsf{P}\langle e\rangle) &\triangleq e \\
\mathtt{event}(\mathcal{E}\langle\tau\rangle) &\ \text{undefined}
\end{aligned}$$

Event-Annotated Operational Semantics

$$\frac{\mathsf{P} \xrightarrow{\mathcal{E}\langle\tau\rangle} \mathsf{P}'}{\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}', M, PB, B, \pi}\ \textsc{A-SilentP}$$

$$\frac{M, PB, B \xrightarrow{\lambda} M', PB', B' \qquad \lambda \in \{\mathsf{B}\langle e\rangle, \mathsf{B}\langle e,-\rangle, \mathsf{P}\langle e\rangle, \mathsf{P}\langle e,-\rangle\} \qquad \mathsf{fresh}(\lambda,\pi)}{\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi.\lambda}\ \textsc{A-PropM}$$

$$\frac{\mathsf{P} \xrightarrow{\lambda} \mathsf{P}' \quad M, PB, B \xrightarrow{\lambda} M', PB', B' \quad \mathsf{fresh}(\lambda,\pi)}{\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}', M', PB', B', \pi.\lambda}\ \textsc{A-Step}$$

where

$$\begin{aligned}\mathsf{fresh}(\lambda,\pi) \triangleq \lambda \notin \pi \land \forall e, w, S.\ \forall w' \neq w.\ \forall S' \neq S.\\ (\lambda{=}\mathsf{R}\langle e,w\rangle \Rightarrow \mathsf{R}\langle e,w'\rangle \notin \pi) \land (\lambda{=}\mathsf{U}\langle e,w\rangle \Rightarrow \mathsf{U}\langle e,w'\rangle \notin \pi)\\ (\lambda{=}\mathsf{P}\langle e,S\rangle \Rightarrow \mathsf{P}\langle e,S'\rangle \notin \pi) \land (\lambda{=}\mathsf{B}\langle e,S\rangle \Rightarrow \mathsf{B}\langle e,S'\rangle \notin \pi)\end{aligned}$$

Definition 11.

$$\mathtt{getE}(.) : \text{ALabels} \overset{\text{fin}}{\rightharpoonup} E$$

$$\mathtt{getE}(\lambda) \triangleq \begin{cases} e & \text{if } \lambda \in \left\{\mathsf{R}\langle e,-\rangle, \mathsf{U}\langle e,-\rangle, \mathsf{W}\langle e\rangle, \mathsf{NTW}\langle e\rangle, \mathsf{MF}\langle e\rangle, \mathsf{SF}\langle e\rangle, \mathsf{FO}\langle e\rangle, \mathsf{FL}\langle e\rangle\right\}\\ \text{undef} & \text{otherwise}\end{cases}$$

$$\mathtt{getVE}(.) : \text{ALabels} \overset{\text{fin}}{\rightharpoonup} E$$

$$\mathtt{getVE}(\lambda) \triangleq \begin{cases} e & \text{if } \lambda \in \left\{\mathsf{R}\langle e,-\rangle, \mathsf{U}\langle e,-\rangle, \mathsf{MF}\langle e\rangle, \mathsf{B}\langle e\rangle, \mathsf{B}\langle e,-\rangle\right\}\\ e & \text{if } \lambda \in \left\{\mathsf{P}\langle e\rangle \,\middle|\, e \in \mathit{NTW} \cup W_{\mathsf{nc}} \cup W_{\mathsf{wt}}\right\}\\ e & \text{if } \lambda \in \left\{\mathsf{P}\langle e,-\rangle \,\middle|\, e \in \mathit{FL}\right\}\\ \text{undef} & \text{otherwise}\end{cases}$$

$$\mathtt{getPE}(.) : \text{ALabels} \overset{\text{fin}}{\rightharpoonup} E$$

$$\mathtt{getPE}(\lambda) \triangleq \begin{cases} e & \text{if } \lambda \in \left\{\mathsf{U}\langle e,-\rangle \,\middle|\, \mathtt{loc}(e) \notin \text{Loc}_{\mathsf{wb}}\right\}\\ e & \text{if } \lambda \in \left\{\mathsf{P}\langle e\rangle, \mathsf{P}\langle e,-\rangle\right\}\\ \text{undef} & \text{otherwise}\end{cases}$$

Definition 12.

$$\begin{aligned}
\mathsf{wfp}(\pi) \triangleq\ & \forall \lambda, \pi_1, \pi_2, e, r, u, e_1, e_2, \lambda_1, \lambda_2, x, y, S.\\
& \mathsf{nodups}(\pi) \land \forall \lambda \in \pi.\ \mathtt{tid}(\mathtt{getE}(\lambda)) \neq 0\\
& \pi{=}\pi_1.\mathsf{R}\langle r,e\rangle.\pi_2 \lor \pi{=}\pi_1.\mathsf{U}\langle u,e\rangle.\pi_2 \Rightarrow \mathsf{wfrd}(r,e,\pi_1)\\
& \pi{=}\pi_1.\mathsf{P}\langle e,S\rangle.\pi_2 \land e \in \mathit{FL} \Rightarrow \forall w \in S.\ \mathsf{wffl}(e,w,\pi_1)\\
& \pi{=}\pi_1.\mathsf{B}\langle e,S\rangle.\pi_2 \land e \in \mathit{FO} \Rightarrow \forall w \in S.\ \mathsf{wffo}(e,w,\pi_1)\\
& \pi{=}\pi_1.\mathsf{P}\langle e,w\rangle.\pi_2 \land e \in \mathit{FO} \Rightarrow \mathsf{wfpfo}(e,w,\pi_1)\\
& \lambda \in \pi \land \mathtt{getVE}(\lambda) = e \Rightarrow \exists! \lambda'.\ \lambda' \leqslant_\pi \lambda \land \mathtt{getE}(\lambda') = e\\
& \lambda \in \pi \land \mathtt{getPE}(\lambda) = e \Rightarrow \exists! \lambda'.\ \lambda' \leqslant_\pi \lambda \land \mathtt{getVE}(\lambda') = e\\
& (e_1,e_2) \in \mathsf{PPO}(\pi) \land \lambda_2 \in \pi \land \mathtt{getVE}(\lambda_2){=}e_2 \Rightarrow \exists! \lambda_1.\ \lambda_1 <_\pi \lambda_2 \land \mathtt{getVE}(\lambda_1){=}e_1\\
& \lambda \in \pi \land \lambda{=}\mathsf{P}\langle e,w\rangle \land e \in \mathit{FO} \Rightarrow \exists S.\ w \in S \land \mathsf{B}\langle e,S\rangle <_\pi \lambda\\
& \left(\begin{array}{l} e_1, e_2 \in \mathit{ST}_x \land \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2 \land \lambda \in \pi \land \mathtt{getPE}(\lambda) = e_2\\ \qquad\qquad \Rightarrow \exists \lambda'.\ \mathtt{getPE}(\lambda') = e_1 \land \lambda' <_\pi \lambda \end{array}\right)\\
& \left(\begin{array}{l} x, y \in \text{Loc}_{\mathsf{wb}} \land (x,y) \in \mathsf{scl} \land e_1 \in \mathit{ST}_x \land e_2 \in \mathit{FL}_y\\ \land\ \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2\\ \qquad\qquad \Rightarrow \exists \lambda'.\ \mathtt{getPE}(\lambda') = e_1 \land \lambda' <_\pi \lambda_2 \end{array}\right)\\
& \left(\begin{array}{l} x, y \in \text{Loc}_{\mathsf{wb}} \land (x,y) \in \mathsf{scl} \land e_1, e \in \mathit{ST}_x \land e_2 \in \mathit{FO}_y\\ \land\ \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2 \land \lambda{=}\mathsf{P}\langle e_2,e\rangle \land \lambda \in \pi\\ \qquad\qquad \Rightarrow \exists \lambda'.\ \mathtt{getPE}(\lambda') = e_1 \land \lambda' <_\pi \lambda \end{array}\right)
\end{aligned}$$

$$\begin{pmatrix} x, y \in \mathrm{Loc}_{\mathsf{wb}} \land (x, y) \in \mathsf{scl} \land e_1 \in FO_y \land e_2 \in ST_x \\ \land\, \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2 \land e_2{=}\mathtt{getPE}(\lambda) \land \lambda \in \pi \\ \qquad \Rightarrow \exists e \in ST_x.\ \mathsf{P}\langle e_1, e\rangle <_\pi \lambda \end{pmatrix}$$

$$\begin{pmatrix} e_1, e_2 \in FO \land (\mathtt{loc}(e_1), \mathtt{loc}(e_2)) \in \mathsf{scl} \land \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \\ \land\, \lambda_1 <_\pi \lambda_2 \land \mathsf{P}\langle e_2, e\rangle \in \pi \\ \qquad \Rightarrow \exists e' \in ST_{\mathtt{loc}(e)}.\ \mathsf{P}\langle e_1, e'\rangle <_\pi \mathsf{P}\langle e_2, e\rangle \end{pmatrix}$$

$$\begin{pmatrix} e_1 \in FO \land e_2 \in FL \land (\mathtt{loc}(e_1), \mathtt{loc}(e_2)) \in \mathsf{scl} \land \lambda_1{=}\mathsf{B}\langle e_1, S\rangle \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2 \\ \qquad \Rightarrow \forall e' \in S.\ \mathsf{P}\langle e_1, e'\rangle <_\pi \lambda_2 \end{pmatrix}$$

$$\begin{pmatrix} e_1 \in FO \land e_2 \in MF \cup SF \cup U \land \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \land \mathsf{B}\langle e_1, S\rangle <_\pi \lambda_2 \land \mathtt{getVE}(\lambda_2){=}e_2 \\ \qquad \Rightarrow \forall w \in S.\ \mathsf{P}\langle e_1, w\rangle <_\pi \lambda_2. \end{pmatrix}$$

where

$$\mathsf{nodups}(\pi) \triangleq \forall \pi_1, \pi_2, \lambda.\ \pi = \pi_1.\lambda.\pi_2 \Rightarrow \mathsf{fresh}(\lambda, \pi_1.\pi_2)$$

$$\mathsf{PO}(\pi) \triangleq \left\{(e_1, e_2) \,\middle|\, \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \land \exists \lambda_1, \lambda_2.\ \lambda_1 <_\pi \lambda_2 \land \mathtt{getE}(\lambda_1){=}e_1 \land \mathtt{getE}(\lambda_2){=}e_2\right\}$$

$$\mathsf{PPO}(\pi) \triangleq \mathsf{ppo}(\mathsf{PO}(\pi))$$

$$\mathsf{wffl}(e, w, \pi) \overset{\text{def}}{\Leftrightarrow} \mathtt{pread}(\pi, \mathtt{loc}(w)){=}w$$

$$\mathsf{wffo}(e, w, \pi) \overset{\text{def}}{\Leftrightarrow} \mathtt{vread}(\pi, \mathtt{loc}(w)){=}w$$

$$\mathsf{wfpfo}(e, w, \pi) \overset{\text{def}}{\Leftrightarrow} \mathtt{pread}(\pi, \mathtt{loc}(w)){=}w$$

$$\mathsf{wfrd}(r, w, \pi) \overset{\text{def}}{\Leftrightarrow} \mathtt{lread}(\pi, \mathtt{loc}(r), \mathtt{tid}(r)){=}w$$

$$\mathtt{pread}(\pi, x) \triangleq \begin{cases} e & \text{if } \exists \pi_1, \pi_2, \lambda.\ e \in ST_x \land \pi{=}\pi_1.\lambda.\pi_2 \land \mathtt{getPE}(\lambda){=}e \\ & \quad \land \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \mathtt{getPE}(\lambda'){=}e'\right\} = \emptyset \\ init_x & \text{otherwise} \end{cases}$$

$$\mathtt{vread}(\pi, x) \triangleq \begin{cases} e & \text{if } \exists \pi_1, \pi_2, \lambda.\ e \in ST_x \land \pi{=}\pi_1.\lambda.\pi_2 \land \mathtt{getVE}(\lambda){=}e \\ & \quad \land \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \mathtt{getVE}(\lambda'){=}e'\right\} = \emptyset \\ init_x & \text{otherwise} \end{cases}$$

$$\mathtt{lread}(\pi, x, \tau) \triangleq \begin{cases} e & \text{if } \exists \pi_1, \pi_2, \lambda.\ e \in ST_x \land \pi{=}\pi_1.\lambda.\pi_2 \land \mathtt{getE}(\lambda){=}e \land \mathtt{tid}(e){=}\tau \\ & \quad \land \forall \lambda' \in \pi.\ \mathtt{getVE}(\lambda') \neq e \\ & \quad \land \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \mathtt{getE}(\lambda'){=}e' \land \mathtt{tid}(e'){=}\tau\right\} = \emptyset \\ \mathtt{vread}(\pi, x) & \text{otherwise} \end{cases}$$

Proposition 1. *For all $\pi, \pi' \in$ PATH, if $\mathsf{wfp}(\pi)$ holds then:*

- $\forall \pi'.\ \pi{=}\pi'.- \Rightarrow \mathsf{wfp}(\pi')$
- $\mathsf{PO}(\pi) \subseteq \mathsf{PO}(\pi.\pi')$
- $\mathsf{PPO}(\pi) \subseteq \mathsf{PPO}(\pi.\pi')$

Definition 13.

$$\mathsf{wf}(M, PB, B, \pi) \overset{\text{def}}{\Leftrightarrow} \mathsf{mem}(\pi) = M \land \forall x \in \mathrm{Loc}_{\mathsf{wb}}.\ PB(x){=}\mathtt{pbuff}(\pi, x) \land \forall \tau.\ B(\tau){=}\mathtt{buff}(\pi, \tau) \land \mathsf{wfp}(\pi)$$

where

$$\texttt{mem}(\pi) = M \overset{\text{def}}{\Leftrightarrow} \forall x \in \text{Loc}.\ M(x) = \texttt{pread}(\pi, x)$$

$$\texttt{pbuff}(\epsilon, x) \triangleq \epsilon$$

$$\texttt{pbuff}(\lambda.\pi, x) \triangleq \begin{cases} e.\texttt{pbuff}(\pi, x) & \text{if } e{=}\texttt{getVE}(\lambda) \wedge e{\in}\text{PBE}\textsc{vent}_x \cap ST \wedge \mathsf{P}\langle e \rangle \notin \pi \\ & \text{if } \lambda{=}\mathsf{B}\langle e, - \rangle \wedge \forall w.\ \texttt{loc}(w){=}x \Rightarrow \mathsf{P}\langle e, w \rangle \notin \pi \\ \texttt{pbuff}(\pi, x) & \text{otherwise} \end{cases}$$

$$\texttt{buff}(\epsilon, \tau) \triangleq \epsilon$$

$$\texttt{buff}(\lambda.\pi, \tau) \triangleq \begin{cases} e.\texttt{buff}(\pi, \tau) & \text{if } \texttt{getE}(\lambda){=}e \wedge e \in \text{BE}\textsc{vent} \wedge \texttt{tid}(e){=}\tau \wedge \forall \lambda' \in \pi.\ \texttt{getVE}(\lambda') \neq e \\ \texttt{buff}(\pi, \tau) & \text{otherwise} \end{cases}$$

Proposition 2. *For all $M, PB, B, \pi, \pi', \tau, x$, if $\texttt{wf}(M, PB, B, \pi)$, then:*

- $\texttt{PO}(B(\tau)) \subseteq \texttt{PO}(\pi)$
- $\texttt{PPO}(B(\tau)) \subseteq \texttt{PPO}(\pi)$
- $M(x) = \texttt{pread}(\pi, x)$
- $\texttt{rd}(M, PB(x), \epsilon, x) = \texttt{vread}(\pi, x)$
- $\texttt{rd}(M, PB(x), B(\tau), x) = \texttt{lread}(\pi, x, \tau)$

Let $B_0 \triangleq \lambda\tau.\epsilon$, $PB_0 \triangleq \lambda x.\epsilon$ and $M_0 \triangleq \lambda x.init_x$ with $\texttt{lab}(init_x) \triangleq (\texttt{W}, x, 0)$ and $\texttt{tid}(init_x){=}0$.

Lemma 1. *For all $\mathsf{P}, \mathsf{P}', PB, PB', B, B', \pi, \pi'$:*

- $\texttt{wf}(M_0, PB_0, B_0, \epsilon)$
- *if* $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}', M', PB', B', \pi'$ *and* $\texttt{wf}(M, PB, B, \pi)$, *then* $\texttt{wf}(M', PB', B', \pi')$
- *if* $\mathsf{P}, M_0, PB_0, B_0, \epsilon \Rightarrow^* \mathsf{P}', M, PB, B, \pi$, *then* $\texttt{wf}(M, PB, B, \pi)$

Proof. The proof of the first part follows trivially from the definitions of M_0, PB_0, and B_0. The second part follows straightforwardly by induction on the structure of $\Rightarrow$. The last part follows from the previous two parts and induction on the length of $\Rightarrow^*$. □

Definition 14.

$$\texttt{getG}(\pi) \triangleq \begin{cases} (E, P, \text{po}, \text{rf}, \text{mo}, \text{pf}) & \text{if wfp}(\pi) \\ \text{undefined} & \text{otherwise} \end{cases}$$

where:

$$E \triangleq E^0 \cup \left\{ e \;\middle|\; \exists \lambda \in \pi.\ \texttt{getVE}(\lambda) = e \wedge \forall e'.\ (e', e) \in \texttt{PO}(\pi) \Rightarrow \exists \lambda' \in \pi.\ \texttt{getVE}(\lambda') = e' \right\}$$

$$E^0 \triangleq \left\{ init_x \;\middle|\; x \in \text{Loc} \right\}$$

$$P(x) \triangleq \texttt{pread}(\pi, x) \qquad \text{for all } x \in \text{Loc}$$

$$\text{rf} \triangleq \texttt{RF}(\pi)|_E \quad \text{with} \quad \texttt{RF}(\pi) \triangleq \left\{ (w, e) \;\middle|\; \mathsf{R}\langle e, w \rangle \in \pi \vee \mathsf{U}\langle e, w \rangle \in \pi \right\}$$

$$\text{po} \triangleq \texttt{PO}(\pi)|_E \cup E^0 \times (E \setminus E^0)$$

$$\text{mo} \triangleq \texttt{MO}(\pi)|_E \cup E^0 \times (E \setminus E^0) \quad \text{with:}$$

$$\texttt{MO}(\pi) \triangleq \left\{ (e_1, e_2) \in \text{sloc} \cap ((E \cap ST) \times (E \cap ST)) \;\middle|\; \begin{array}{l} \exists \lambda_1, \lambda_2.\ e_1{=}\texttt{getVE}(\lambda_1) \wedge e_2{=}\texttt{getVE}(\lambda_2) \\ \wedge\ \lambda_1 <_\pi \lambda_2 \end{array} \right\}$$

$$\text{pf} \triangleq \texttt{PF}(\pi)|_E \quad \text{with} \quad \texttt{PF}(\pi) \triangleq \left\{ (w, e) \;\middle|\; \exists S.\ w \in S \wedge (\mathsf{P}\langle e, S \rangle \in \pi \vee \mathsf{B}\langle e, S \rangle \in \pi) \right\}$$

Soundness of the Event-Annotated Semantics against PEx86 Declarative Semantics

Lemma 2. *For all* π *and* $G = (E, P, \mathrm{po}, \mathrm{rf}, \mathrm{mo}, \mathrm{pf})$, *if* $\texttt{getG}(\pi) = G$, *then:*

(1) $\mathrm{mo}_i \subseteq \mathrm{po}$
(2) $\mathrm{rf}_i \subseteq \mathrm{po}$
(3) $\mathrm{rb}_i \subseteq \mathrm{po}$

Proof. Pick arbitrary π and $G = (E, P, \mathrm{po}, \mathrm{rf}, \mathrm{mo}, \mathrm{pf})$ such that $\texttt{getG}(\pi) = G$. From the definition of $\texttt{getG}(.)$ we then know $\mathrm{wfp}(\pi)$ holds. We prove each part separately. In what follows we write $\lambda \preccurlyeq_\pi \lambda'$ as a shorthand for $\lambda \prec_\pi \lambda' \lor \lambda{=}\lambda'$.

RTS (1)
Pick arbitrary e_1, e_2 such that $(e_1, e_2) \in \mathrm{mo}_i$. From the definition of mo we then know that either i) $(e_1, e_2) \in E^0 \times (E \setminus E^0)$; or ii) $(e_1, e_2)\texttt{MO}(\pi)|_E$. In case (i) from the definition of po we then have $(e_1, e_2) \in \mathrm{po}$, as required.

In case (ii), from the definition of $\texttt{MO}(\pi)$ we know $e_1, e_2 \in ST$, $(e_1, e_2) \in \mathrm{sloc}$, and there exist λ_1, λ_2 such that $\texttt{getVE}(\lambda_1){=}e_1$, $\texttt{getVE}(\lambda_2){=}e_2$ and$\lambda_1 \prec_\pi \lambda_2$. Moreover, as $\mathrm{wfp}(\pi)$ holds, we know there exist unique $\lambda'_1, \lambda'_2 \in \pi$ such that $\texttt{getE}(\lambda'_1){=}e_1$, $\lambda'_1 \preccurlyeq_\pi \lambda_1$, $\texttt{getE}(\lambda'_2){=}e_2$ and $\lambda'_2 \preccurlyeq_\pi \lambda_2$. There are now two cases to consider: a) $\lambda'_1 \prec_\pi \lambda'_2$; or b) $\lambda'_2 \prec_\pi \lambda'_1$.

In case (a), from the definition of $\texttt{PO}(\pi)$ we have $(e_1, e_2) \in \texttt{PO}(\pi)$ and thus $(e_1, e_2) \in \mathrm{po}$, as required. In case (b), from the definition of $\texttt{PO}(\pi)$ we have $(e_2, e_1) \in \texttt{PO}(\pi)$ and thus $(e_1, e_2) \in \mathrm{po}$. As such, since $e_1, e_2 \in ST$, $(e_1, e_2) \in \mathrm{sloc}$ and $(e_2, e_1) \in \mathrm{po}$, given the definition of $\mathrm{ppo}(.)$ we have $(e_2, e_1) \in \mathrm{ppo}(\texttt{PO}(\pi))$ and thus $(e_2, e_1) \in \texttt{PPO}(\pi)$. As such, since $\lambda_1 \in \pi$, $\texttt{getVE}(\lambda_1){=}e_1$ and $\mathrm{wfp}(\pi)$ holds, from the definition of $\mathrm{wfp}(.)$ we know there exists a unique λ''_2 such that $\lambda''_2 \prec_\pi \lambda_1$ and $\texttt{getVE}(\lambda''_2){=}e_2$. Consequently, as $\texttt{getVE}(\lambda''_2){=}e_2$, $\texttt{getVE}(\lambda_2){=}e_2$ and λ''_2 is unique in π, we know $\lambda''_2{=}\lambda_2$. Therefore, as $\lambda''_2 \prec_\pi \lambda_1$, we have $\lambda_2 \prec_\pi \lambda_1$. This however leads to a contradiction as we also have $\lambda_1 \prec_\pi \lambda_2$ and $\prec_\pi$ is a strict total order.

RTS (2)
Pick arbitrary w, r such that $(w, r) \in \mathrm{rf}_i$ and thus $w, r \in E$. Let $\texttt{tid}(w) = \texttt{tid}(r) = \tau$. From the definition of rf we then know there exist π_1, π_2, λ such that $\pi{=}\pi_1.\lambda.\pi_2$ and $\lambda{=}\mathsf{R}\langle r, w\rangle$ or $\lambda{=}\mathsf{U}\langle r, w\rangle$. As such, we have $\texttt{getE}(\lambda){=}r$. As $\mathrm{wfp}(\pi)$ holds and $\pi{=}\pi_1.\lambda.\pi_2$, we then have $\mathrm{wfrd}(r, w, \pi_1)$. From the definition of $\mathrm{wfrd}(r, w, \pi_1)$ and since $\texttt{tid}(w) = \texttt{tid}(r) = \tau$, we then know that there exists λ' such that $\pi_1{=} -.\lambda'.-$ and either i) $\texttt{getE}(\lambda'){=}w$; or ii) $\texttt{getVE}(\lambda'){=}w$.

In both cases, as $\pi{=}\pi_1.\lambda.\pi_2$ and $\pi_1{=}-.\lambda'.-$, we have $\lambda' \prec'_\pi \lambda$. In case (i), as $\texttt{getE}(\lambda){=}r$, $\texttt{getE}(\lambda'){=}w$, $\lambda' \prec'_\pi \lambda$ and $w, r \in E$, from the definition of po we have $(w, r) \in \mathrm{po}$, as required. In case (ii), since $\texttt{getVE}(\lambda'){=}w$ and $\mathrm{wfp}(\pi)$, we know there exists λ'' such that $\texttt{getE}(\lambda''){=}w$ and $\lambda'' \preccurlyeq_\pi \lambda'$. As such, since $\lambda' \prec'_\pi \lambda$, from the transitivity of $\prec_\pi$ we have $\lambda'' \prec_\pi \lambda$. Consequently, since $\texttt{getE}(\lambda){=}r$, $\texttt{getE}(\lambda''){=}w$, $\lambda'' \prec'_\pi \lambda$ and $w, r \in E$, from the definition of po we have $(w, r) \in \mathrm{po}$, as required.

RTS (3)
Pick arbitrary r, w such that $(r, w) \in \mathrm{rb}_i$. That is, there exist w', τ, x such that $(w', r) \in \mathrm{rf}$, $(w', w) \in \mathrm{mo}$, $\texttt{loc}(w'){=}\texttt{loc}(r){=}\texttt{loc}(r){=}x$, $w, w', r \in E$, $(w, w') \in ST_x$ and $\texttt{tid}(w) = \texttt{tid}(r) = \tau$. As $(w', r) \in \mathrm{rf}$, we know there exist π_a, π_b, λ_r such that $\pi{=}\pi_a.\lambda_r.\pi_b$ and $\lambda_r{=}\mathsf{R}\langle r, w'\rangle$ or $\lambda_r{=}\mathsf{U}\langle r, w'\rangle$ and thus $\texttt{getE}(\lambda_r){=}\texttt{getVE}(\lambda_r){=}r$. As $w \in E$ and $\mathrm{wfp}(\pi)$ holds, we know there exist $\lambda_w \in \pi$ such that $\texttt{getE}(\lambda_w){=}w$. There are two cases to consider: i) $\lambda_w \in \pi_b$; or ii) $\lambda_w \in \pi_a$. In case (i), we then have $\lambda_r \prec_\pi \lambda_w$. As such, since $\texttt{getE}(\lambda_r){=}r$, $\texttt{getE}(\lambda_w){=}w$, and $\texttt{tid}(w) = \texttt{tid}(r) = \tau$, from the definition of po we have $(r, w) \in \mathrm{po}$, as required. In case (ii), we proceed by contradiction.

As $\mathsf{wfp}(\pi)$ and thus $\mathsf{wfrd}(r, w', \pi_a)$ holds, there are three cases to consider:

Case (A)
There exist $\pi_1, \pi_2, \lambda^v_{w'}$ such that $\pi_a{=}\pi_1.\lambda^v_{w'}.\pi_2$, $\mathtt{getVE}(\lambda^v_{w'}){=}w'$, $\left\{\lambda' \in \pi_2 \,\middle|\, \mathtt{getVE}(\lambda') \in ST_x\right\} = \emptyset$ and $\left\{\lambda' \in \pi_a \,\middle|\, \exists e' \in ST_x.\, \mathtt{getE}(\lambda') = e' \land \mathtt{tid}(e') = \mathtt{tid}(r) \land \forall \lambda'' \in \pi_a.\, \mathtt{getVE}(\lambda'') \neq e'\right\} = \emptyset$. As such, since from the assumption of case (ii) we have $\lambda_w \in \pi_a$, $\mathtt{getE}(\lambda_w){=}w$, $w \in ST_x$ and $\mathtt{tid}(w){=}\mathtt{tid}(r)$, from the last two constraints we know there exists $\lambda^v_w \in \pi_1$ such that $\mathtt{getVE}(\lambda^v_w){=}w$. That is, $\lambda^v_w <_\pi \lambda^v_{w'}$. On the other hand, as $(w', w) \in \mathsf{mo}$, $\mathtt{getVE}(\lambda^v_{w'}){=}w'$ and $\mathtt{getVE}(\lambda^v_w){=}w$, from the definition of mo and the uniqueness of events in π (which follows from $\mathsf{wfp}(\pi)$) we have $\lambda^v_{w'} <_\pi \lambda^v_w$, leading to a contradiction as we also have $\lambda^v_w <_\pi \lambda^v_{w'}$ and $<_\pi$ is a strict total order.

Case (B)
There exist $\pi_1, \pi_2, \lambda_{w'}$ such that $\pi_a{=}\pi_1.\lambda_{w'}.\pi_2$, $\mathtt{tid}(w'){=}\mathtt{tid}(r){=}\mathtt{tid}(w)$, $\mathtt{getE}(\lambda_{w'}){=}w'$, $\forall \lambda' \in \pi_a.\, \mathtt{getVE}(\lambda') \neq w'$ and $\left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\, \mathtt{getE}(\lambda') = e' \land \mathtt{tid}(e') = \mathtt{tid}(r)\right\} = \emptyset$. As such, since from the assumption of case (ii) we have $\lambda_w \in \pi_a$, $\mathtt{getE}(\lambda_w){=}w$, $w \in ST_x$ and $\mathtt{tid}(w){=}\mathtt{tid}(r)$, from the last constraint we know there exists $\lambda_w \in \pi_1$. That is, $\lambda_w <_\pi \lambda_{w'}$, and thus since $\mathtt{getE}(\lambda_{w'}){=}w'$, $\mathtt{getE}(\lambda_w){=}w$, $\mathtt{tid}(w){=}\mathtt{tid}(w')$, we also have $(w, w') \in \mathsf{PO}(\pi)$. As such, from the definition of $\mathsf{ppo}(.)$ and since $w, w' \in ST_x$ we have $(w, w') \in \mathsf{ppo}(\mathsf{PO}(\pi))$. On the other hand, since $(w', w) \in mo$, we know there exist $\lambda^v_w, \lambda^v_{w'}$ such that $\mathtt{getVE}(\lambda^v_w){=}w$, $\mathtt{getVE}(\lambda^v_{w'}){=}w'$ and $\lambda^v_{w'} <_\pi \lambda^v_w$. However, since $(w, w') \in \mathsf{ppo}(\mathsf{PO}(\pi))$, from $\mathsf{wfp}(\pi)$ (and the uniqueness of events guaranteed by it) we have $\lambda^v_w <_\pi \lambda^v_{w'}$, leading to a contradiction as $<_\pi$ is a strict total order.

Case (C)
$w' = init_x$ and $\left\{\lambda \in \pi_a \,\middle|\, \exists e' \in ST_x.\, \mathtt{getVE}(\lambda){=}e' \lor (\mathtt{getE}(\lambda){=}e' \land \mathtt{tid}(e'){=}\mathtt{tid}(r))\right\} = \emptyset$. As $\mathtt{getE}(\lambda_w){=}w$, $w \in ST_x$ and $\mathtt{tid}(w){=}\mathtt{tid}(r)$, this last constraint asserts that $\lambda_w \notin \pi_a$, contradicting the assumption of case (ii). □

Lemma 3. *For all π and $G = (E, P, \mathrm{po}, \mathsf{rf}, \mathsf{mo}, \mathsf{pf})$, if $\mathtt{getG}(\pi) = G$, then:*

(1) $(e_1, e_2) \in \mathsf{ppo}(\mathrm{po}) \Rightarrow (e_1 \in E^0 \land e_2 \in E \setminus E^0) \lor \exists \lambda_1, \lambda_2.\, \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2$
(2) $(e_1, e_2) \in \mathsf{mo} \Rightarrow (e_1 \in E^0 \land e_2 \in E \setminus E^0) \lor \exists \lambda_1, \lambda_2.\, \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2$
(3) $(e_1, e_2) \in \mathsf{rf}_e \Rightarrow (e_1 \in E^0 \land e_2 \in E \setminus E^0) \lor \exists \lambda_1, \lambda_2.\, \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2$
(4) $(e_1, e_2) \in \mathsf{rb}_e \Rightarrow \exists \lambda_1, \lambda_2.\, \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2$
(5) $(e_1, e_2) \in \mathsf{pf} \Rightarrow (e_1 \in E^0 \land e_2 \in E \setminus E^0) \lor \exists \lambda_1, \lambda_2.\, \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2$
(6) $(e_1, e_2) \in \mathsf{pb} \Rightarrow \exists \lambda_1, \lambda_2.\, \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2$
(7) $(e_1, e_2) \in \mathsf{ob} \Rightarrow (e_1 \in E^0 \land e_2 \in E \setminus E^0) \lor \exists \lambda_1, \lambda_2.\, \mathtt{getVE}(\lambda_1){=}e_1 \land \mathtt{getVE}(\lambda_2){=}e_2 \land \lambda_1 <_\pi \lambda_2$
(8) $\mathsf{irreflexive}(\mathsf{ob})$

Proof. Pick arbitrary π and $G = (E, P, \mathrm{po}, \mathsf{rf}, \mathsf{mo}, \mathsf{pf})$ such that $\mathtt{getG}(\pi) = G$. From the definition of $\mathtt{getG}(.)$ we then know $\mathsf{wfp}(\pi)$ holds. We prove each part separately. In what follows we write $\lambda \leqslant_\pi \lambda'$ as a shorthand for $\lambda <_\pi \lambda' \lor \lambda{=}\lambda'$.

RTS (1)
Pick arbitrary e_1, e_2 such that $(e_1, e_2) \in \mathsf{ppo}(\mathrm{po})$. From the definitions of po, ppo we then know either i) $e_1 \in E^0 \land e_2 \in E \setminus E^0$; or ii) $(e_1, e_2) \in \mathsf{PO}(\pi)|_E$ and $(e_1, e_2) \in \mathsf{PPO}(\pi)|_E$. In case (i) the desired result follows immediately. In case (ii), as $(e_1, e_2) \in \mathsf{PPO}(\pi)|_E$, we know $e_2 \in E$ and thus there exists $\lambda_2 \in \pi$ such that $\mathtt{getVE}(\lambda_2){=}e_2$. Consequently, as $(e_1, e_2) \in \mathsf{PPO}(\pi)|_E$ and thus $(e_1, e_2) \in \mathsf{PPO}(\pi)$, $\lambda_2 \in \pi$ and $\mathtt{getVE}(\lambda_2){=}e_2$, from $\mathsf{wfp}(\pi)$ we know there exists λ_1 such that $\mathtt{getVE}(\lambda_1){=}e_1$ and

$\lambda_1 <_\pi \lambda_2$, as required.

RTS (2)
Pick arbitrary e_1, e_2 such that $(e_1, e_2) \in \texttt{mo}$. From the definition of mo we know either i) $e_1 \in E^0 \land e_2 \in E \setminus E^0$; or ii) $(e_1, e_2) \in \texttt{MO}(\pi)|_E$. In case (i) the desired result follows immediately. In case (ii), from the definition of $\texttt{MO}(.)$ we immediately know there exist λ_1, λ_2 such that $\texttt{getVE}(\lambda_1)=e_1$, $\texttt{getVE}(\lambda_2)=e_2$ and $\lambda_1 <_\pi \lambda_2$, as required.

RTS (3)
Pick arbitrary w, r such that $(w, r) \in \texttt{rf}_e$ and thus $w, r \in E$, and $\texttt{tid}(w) \neq \texttt{tid}(r)$. Let $\texttt{loc}(w) = \texttt{loc}(r) = x$. From the definition of rf we then know there exist π_1, π_2, λ such that $\pi{=}\pi_1.\lambda.\pi_2$ and $\lambda{=}\texttt{R}\langle r, w\rangle$ or $\lambda{=}\texttt{U}\langle r, w\rangle$ and thus (from $\texttt{wfp}(\pi)$ and the definition of E^0) we have $r \in E \setminus E^0$. As such, we have $\texttt{getE}(\lambda){=}\texttt{getVE}(\lambda){=}r$. As $\texttt{wfp}(\pi)$ holds and $\pi{=}\pi_1.\lambda.\pi_2$, we then have $\texttt{wfrd}(r, w, \pi_1)$. From the definition of $\texttt{wfrd}(r, w, \pi_1)$ and since $\texttt{tid}(w) \neq \texttt{tid}(r)$, there are two cases to consider: i) $w{=}init_x$ and thus $w \in E^0$; or ii) there exists λ' such that $\pi_1{=}-.\lambda'.-$ and $\texttt{getVE}(\lambda'){=}w$. In case (i) the desired result holds immediately as we have $r \in E \setminus E^0$ and $w \in E^0$. In case (ii), as $\pi{=}\pi_1.\lambda.\pi_2$ and $\pi_1{=}-.\lambda'.-$, we have $\lambda' <_\pi \lambda$. As such, we have $\texttt{getVE}(\lambda){=}r$, $\texttt{getVE}(\lambda'){=}w$ and $\lambda' <_\pi \lambda$, as required.

RTS (4)
Pick arbitrary r, w such that $(r, w) \in \texttt{rb}_e$. That is, there exist w', x such that $(w', r) \in \texttt{rf}$, $(w', w) \in \texttt{mo}$, $\texttt{loc}(w'){=}\texttt{loc}(r){=}\texttt{loc}(r){=}x$, $w, w', r \in E$, $(w, w') \in ST_x$ and $\texttt{tid}(w) \neq \texttt{tid}(r)$. As $(w', r) \in \texttt{rf}$, we know there exist π_a, π_b, λ_r such that $\pi{=}\pi_a.\lambda_r.\pi_b$ and $\lambda_r{=}\texttt{R}\langle r, w'\rangle$ or $\lambda_r{=}\texttt{U}\langle r, w'\rangle$ and thus $\texttt{getE}(\lambda_r){=}\texttt{getVE}(\lambda_r){=}r$. As $w \in E$ and $\texttt{wfp}(\pi)$ holds, we know there exist $\lambda_w, \lambda^v_w \in \pi$ such that $\texttt{getE}(\lambda_w){=}w$, $\texttt{getVE}(\lambda^v_w) = w$ and $\lambda_w <_\pi \lambda^v_w$. There are two cases to consider: i) $\lambda^v_w \in \pi_b$; or ii) $\lambda^v_w \in \pi_a$. In case (i), we then have $\lambda_r <_\pi \lambda^v_w$. As such, we have $\texttt{getVE}(\lambda_r){=}r$, $\texttt{getVE}(\lambda^v_w){=}w$ and $\lambda_r <_\pi \lambda^v_w$, as required. In case (ii), we proceed by contradiction. As $\texttt{wfp}(\pi)$ and thus $\texttt{wfrd}(r, w', \pi_a)$ holds, there are three cases to consider:

Case (A)
There exist $\pi_1, \pi_2, \lambda^v_{w'}$ such that $\pi_a{=}\pi_1.\lambda^v_{w'}.\pi_2$, $\texttt{getVE}(\lambda^v_{w'}){=}w'$, $\left\{\lambda' \in \pi_2 \,\middle|\, \texttt{getVE}(\lambda') \in ST_x\right\} = \emptyset$. As such, since from the assumption of case (ii) we have $\lambda^v_w \in \pi_a$, $\texttt{getVE}(\lambda^v_w){=}w$ and $w \in ST_x$, from the last constraint we know $\lambda^v_w \in \pi_1$. That is, $\lambda^v_w <_\pi \lambda^v_{w'}$. On the other hand, as $(w', w) \in \texttt{mo}$, $\texttt{getVE}(\lambda^v_{w'}){=}w'$ and $\texttt{getVE}(\lambda^v_w){=}w$, from the definition of mo and the uniqueness of events in π (which follows from $\texttt{wfp}(\pi)$) we have $\lambda^v_{w'} <_\pi \lambda^v_w$, leading to a contradiction as we also have $\lambda^v_w <_\pi \lambda^v_{w'}$ and $<_\pi$ is a strict total order.

Case (B)
There exist $\pi_1, \pi_2, \lambda_{w'}$ such that $\pi_a{=}\pi_1.\lambda_{w'}.\pi_2$, $\texttt{tid}(w'){=}\texttt{tid}(r)$, $\texttt{getE}(\lambda_{w'}){=}w'$, $\forall \lambda' \in \pi_a.\ \texttt{getVE}(\lambda') \neq w'$ and $\left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \texttt{getE}(\lambda') = e' \land \texttt{tid}(e') = \texttt{tid}(r)\right\} = \emptyset$. On the other hand, as $(w', w) \in \texttt{mo}$ and $\texttt{getVE}(\lambda^v_w){=}w$, from the definition of mo we know there exists $\lambda^v_{w'}$ such that $\texttt{getVE}(\lambda^v_{w'}){=}w'$ and $\lambda^v_{w'} <_\pi \lambda^v_w$. As such, since from the assumption of case (i) we have $\lambda^v_w \in \pi_a$ and $\lambda^v_{w'} <_\pi \lambda^v_w$, we also have $\lambda^v_{w'} \in \pi_a$. As $\texttt{getVE}(\lambda^v_{w'}){=}w'$ and $\lambda^v_{w'} \in \pi_a$, we arrive at a contradiction since we also have $\forall \lambda' \in \pi_a.\ \texttt{getVE}(\lambda') \neq w'$.

Case (C)
$w' = init_x$ and $\left\{\lambda \in \pi_a \,\middle|\, \exists e' \in ST_x.\ \texttt{getVE}(\lambda){=}e' \lor (\texttt{getE}(\lambda){=}e' \land \texttt{tid}(e'){=}\texttt{tid}(r))\right\} = \emptyset$. As $\texttt{getVE}(\lambda^v_w){=}w$ and $w \in ST_x$, this last constraint asserts that $\lambda_w \notin \pi_a$, contradicting the assumption

of case (ii).

RTS (5)
Pick arbitrary f, w such that $(w, f) \in \mathsf{pf}$ and thus $w, f \in E$. From the definition of pf we then know there exist π_1, π_2, λ, S such that $\pi{=}\pi_1.\lambda.\pi_2$, $w \in S$, and $\lambda{=}\mathsf{P}\langle f, S\rangle$ or $\lambda{=}\mathsf{B}\langle f, S\rangle$, and thus there exists $x, y \in \mathrm{Loc}_{\mathsf{wb}}$ such that $(x, y) \in \mathsf{scl}$, $w \in ST_x$, $f \in FL_y \cup FO_y$ and (from $\mathsf{wfp}(\pi)$ and the definition of E^0) we have $f \in E \setminus E^0$. There are then two cases to consider: A) $f \in FL_y$ and $\lambda{=}\mathsf{P}\langle f, S\rangle$; or B) $f \in FO_y$ and $\lambda{=}\mathsf{B}\langle f, S\rangle$.

Case (A): $f \in FL_y$ and $\lambda{=}\mathsf{P}\langle f, S\rangle$
From the definition of $\mathtt{getPE}(.)$ we have $\mathtt{getPE}(\lambda){=}f$. As $\mathsf{wfp}(\pi)$ holds and $\pi{=}\pi_1.\lambda.\pi_2$, we then have $\mathsf{wffl}(f, w, \pi_1)$. From the definition of $\mathsf{wffl}(f, w, \pi_1)$, there are two cases to consider: i) $w{=}init_x$ and thus $w \in E^0$; or ii) there exists λ' such that $\pi_1{=}-.\lambda'.-$ and $\mathtt{getPE}(\lambda'){=}w$. In case (i) the desired result holds immediately as we have $f \in E \setminus E^0$ and $w \in E^0$.

In case (ii), as $\pi{=}\pi_1.\lambda.\pi_2$ and $\pi_1{=}-.\lambda'.-$, we have $\lambda' \prec_\pi \lambda$. On the other hand, as $\lambda', \lambda \in \pi$, $\mathtt{getPE}(\lambda'){=}w$ and $\mathtt{getPE}(\lambda){=}f$, from $\mathsf{wfp}(\pi)$ we know there exist $\lambda_v, \lambda'_v \in \pi$ such that $\mathtt{getVE}(\lambda_v) = f$ and $\mathtt{getVE}(\lambda'_v) = w$. There are then two cases to consider: a) $\lambda'_v \prec_\pi \lambda_v$; or b) $\lambda_v \prec_\pi \lambda'_v$. In case (a) we have $\mathtt{getVE}(\lambda'_v){=}w$, $\mathtt{getVE}(\lambda_v) = f$ and $\lambda'_v \prec_\pi \lambda_v$, as required. In case (b) since $\lambda_v \prec_\pi \lambda'_v$, $\mathtt{getVE}(\lambda'_v){=}w$, $\mathtt{getVE}(\lambda_v) = f$, $x, y \in \mathrm{Loc}_{\mathsf{wb}}$, $(x, y) \in \mathsf{scl}$, $w \in ST_x$, $f \in FL_y$, $\lambda' \in \pi$ and $\mathtt{getPE}(\lambda'){=}w$, from $\mathsf{wfp}(\pi)$ (and the uniqueness of events it guarantees) we have $\lambda \prec_\pi \lambda'$. This, however, leads to a contradiction as we also have $\lambda' \prec_\pi \lambda$ and $\prec_\pi$ is a strict total order.

Case (B): $f \in FO_y$ and $\lambda{=}\mathsf{B}\langle f, S\rangle$
From the definition of $\mathtt{getVE}(.)$ we have $\mathtt{getVE}(\lambda){=}f$. As $\mathsf{wfp}(\pi)$ holds and $\pi{=}\pi_1.\lambda.\pi_2$, we then have $\mathsf{wffo}(f, w, \pi_1)$. From the definition of $\mathsf{wffo}(f, w, \pi_1)$, there are two cases to consider: i) $w{=}init_x$ and thus $w \in E^0$; or ii) there exists λ' such that $\pi_1{=}-.\lambda'.-$ and $\mathtt{getVE}(\lambda'){=}w$. In case (i) the desired result holds immediately as we have $f \in E \setminus E^0$ and $w \in E^0$. In case (ii) as we have $\pi{=}\pi_1.\lambda.\pi_2$ and $\pi_1{=}-.\lambda'.-$, we also have $\lambda' \prec_\pi \lambda$. Consequently, we have $\mathtt{getVE}(\lambda'){=}w$, $\mathtt{getVE}(\lambda){=}f$ and $\lambda' \prec_\pi \lambda$.

RTS (6)
Pick arbitrary f, w such that $(f, w) \in \mathsf{pb}$ and thus $w, f \in E$. From the definition of pb we then know there exist w' such that $(w', f) \in \mathsf{pf}$ and $(w', w) \in \mathsf{mo}$. Let $\mathtt{loc}(w) = \mathtt{loc}(w') = x$. There are now two cases to consider: A) $f \in FL$; or B) $f \in FO$.

Case (A): $f \in FL$
As $(w', f) \in \mathsf{pf}$, from the definition of pf we know there exists $x, y \in \mathrm{Loc}_{\mathsf{wb}}, \lambda^p_f, S$ such that $w, w' \in ST_x$, $f \in FL_y$, $(x, y) \in \mathsf{scl}$, $w' \in S$, $\pi{=}\pi_1.\lambda^p_f.\pi_2$, $\lambda^p_f{=}\mathsf{P}\langle f, S\rangle$ and thus $\mathtt{getPE}(\lambda^p_f){=}\mathtt{getVE}(\lambda^p_f){=}f$, and (since $\mathsf{wfp}(\pi)$ holds) $\mathsf{wffl}(f, w', \pi_1)$. From the proofs of parts (2) and (5) and since $\mathtt{getVE}(\lambda^p_f){=}f$, we know that $f, w \in E \setminus E^0$ and either i) $w' \in E^0$ and (since $f, w \in E \setminus E^0$) there exist λ_w such that $\mathtt{getVE}(\lambda_w){=}w$; or ii) there exist $\lambda_w, \lambda_{w'}$ such that $\mathtt{getVE}(\lambda_w){=}w$, $\mathtt{getVE}(\lambda_{w'}){=}w'$, $\lambda_{w'} \prec_\pi \lambda^p_f$ and $\lambda_{w'} \prec_\pi \lambda_w$.

In case (i), as $\mathtt{getVE}(\lambda_w){=}w$, $\mathtt{getVE}(\lambda^p_f){=}f$ and $\lambda_w, \lambda^p_f \in \pi$, either $\lambda^p_f \prec_\pi \lambda_w$ or $\lambda_w \prec_\pi \lambda^p_f$. In the former case the desired result follows immediately. In the latter case, since $\mathtt{getVE}(\lambda^p_f){=}f$, $\mathtt{getVE}(\lambda_w){=}w$, $\lambda_w \prec_\pi \lambda^p_f$, $\mathtt{getPE}(\lambda^p_f){=}f$, $\lambda^p_f \in \pi$, $x, y \in \mathrm{Loc}_{\mathsf{wb}}$, $(x, y) \in \mathsf{scl}$, $w \in ST_x$ and $f \in FL_y$, from $\mathsf{wfp}(\pi)$ we know there exist λ^p_w such that $\mathtt{getPE}(\lambda^p_w){=}w$ and $\lambda^p_w \prec_\pi \lambda^p_f$, and thus (since

$\pi{=}\pi_1.\lambda^p_f.\pi_2$) $\lambda^p_w \in \pi_1$. On the other hand, as $w' \in E^0$ and $\texttt{loc}(w') = x$, we know $w' = init_x$. Consequently, from $\text{wffl}(f, w', \pi_1)$ we know $\left\{\lambda \in \pi_1 \mid \exists e' \in ST_x.\ \texttt{getPE}(\lambda) = e'\right\} = \emptyset$. This leads to a contradiction since $w \in ST_x$, $\texttt{getPE}(\lambda^p_w){=}w$ and $\lambda^p_w \in \pi_1$.

In case (ii), as $\texttt{getVE}(\lambda_w){=}w$, $\texttt{getVE}(\lambda^p_f){=}f$, and $\lambda_w, \lambda^p_f \in \pi$, we know either $\lambda^p_f <_\pi \lambda_w$ or $\lambda_w <_\pi \lambda^p_f$. In the former case the desired result follows immediately. In the latter case, since $\texttt{getVE}(\lambda^p_f){=}f$, $\texttt{getVE}(\lambda_w){=}w$, $\lambda_w <_\pi \lambda^p_f$, $\texttt{getPE}(\lambda^p_f){=}f$, $\lambda^p_f \in \pi$, $x, y \in \text{Loc}_{\text{wb}}, \lambda^p_f$, $w \in ST_x$ and $f \in FL_y$, from $\text{wfp}(\pi)$ we know there exist λ^p_w such that $\texttt{getPE}(\lambda^p_w){=}w$ and $\lambda^p_w <_\pi \lambda^p_f$, and thus (since $\pi{=}\pi_1.\lambda^p_f.\pi_2$) $\lambda^p_w \in \pi_1$. Moreover, as $\texttt{getVE}(\lambda_{w'}){=}w'$ and thus from $\text{wfp}(\pi)$ we have $\texttt{tid}(w') \neq 0$, i.e. $w' \notin E^0$, from $\text{wffl}(f, w', \pi_1)$ we know there exist $\pi_a, \pi_b, \lambda^p_{w'}$ such that $\pi_1 = \pi_a.\lambda^p_{w'}.\pi_b$, $\texttt{getPE}(\lambda^p_{w'}) = w'$ and $\left\{\lambda' \in \pi_b \mid \exists e' \in ST_x.\ \texttt{getPE}(\lambda') = e'\right\} = \emptyset$. Consequently, as $w \in ST_x$, $\texttt{getPE}(\lambda^p_w){=}w$, $\lambda^p_w \in \pi_1$ and $\pi_1 = \pi_a.\lambda^p_{w'}.\pi_b$, we have $\lambda^p_w \in \pi_a$. That is, as $\pi_1 = \pi_a.\lambda^p_{w'}.\pi_b$, we have $\lambda^p_w <_\pi \lambda^p_{w'}$. On the other hand, as $x \in \text{Loc}_{\text{wb}}$, $w, w' \in ST_x$, $\texttt{getVE}(\lambda_w){=}w$, $\texttt{getVE}(\lambda_{w'}){=}w'$, $\lambda_{w'} <_\pi \lambda_w$, $\texttt{getPE}(\lambda^p_{w'}) = w'$, $\lambda^p_{w'} \in \pi$ and $\texttt{getPE}(\lambda^p_w){=}w$, from $\text{wfp}(\pi)$ (and the uniqueness of labels it guarantees) we have $\lambda^p_{w'} <_\pi \lambda^p_w$. This, however, leads to a contradiction as we also have $\lambda^p_w <_\pi \lambda^p_{w'}$ and $<_\pi$ is a strict total order.

Case (B): $f \in FO$
As $(w', f) \in \text{pf}$, from the definition of pf we know there exists $x, y \in \text{Loc}_{\text{wb}}, \lambda_f, S$ such that $w, w' \in ST_x$, $f \in FO_y$, $(x, y) \in \text{scl}$, $w' \in S$, $\pi{=}\pi_1.\lambda_f.\pi_2$, $\lambda_f{=}\text{B}\langle f, S\rangle$ and thus $\texttt{getVE}(\lambda_f){=}f$, and (since $\text{wfp}(\pi)$ holds) $\text{wffo}(f, w', \pi_1)$. From the proofs of parts (2) and (5) and since $\texttt{getVE}(\lambda_f){=}f$, we know that $f, w \in E \setminus E^0$ and either i) $w' \in E^0$ and (since $f, w \in E \setminus E^0$) there exist λ_w such that $\texttt{getVE}(\lambda_w){=}w$; or ii) there exist $\lambda_w, \lambda_{w'}$ such that $\texttt{getVE}(\lambda_w){=}w$, $\texttt{getVE}(\lambda_{w'}){=}w'$, $\lambda_{w'} <_\pi \lambda_f$ and $\lambda_{w'} <_\pi \lambda_w$.

In case (i), as $\texttt{getVE}(\lambda_w){=}w$, $\texttt{getVE}(\lambda_f){=}f$ and $\lambda_w, \lambda_f \in \pi$, either $\lambda_f <_\pi \lambda_w$ or $\lambda_w <_\pi \lambda_f$. In the former case the desired result follows immediately. In the latter case, as $\lambda_w <_\pi \lambda_f$, and $\pi{=}\pi_1.\lambda_f.\pi_2$, we have $\lambda_w \in \pi_1$. On the other hand, as $w' \in E^0$ and $\texttt{loc}(w') = x$, we know $w' = init_x$. Consequently, from $\text{wffo}(f, w', \pi_1)$ we know $\left\{\lambda \in \pi_1 \mid \exists e' \in ST_x.\ \texttt{getVE}(\lambda) = e'\right\} = \emptyset$. This leads to a contradiction since $w \in ST_x$, $\texttt{getVE}(\lambda_w){=}w$ and $\lambda_w \in \pi_1$.

In case (ii), as $\texttt{getVE}(\lambda_w){=}w$, $\texttt{getVE}(\lambda_f){=}f$, and $\lambda_w, \lambda_f \in \pi$, we know either $\lambda_f <_\pi \lambda_w$ or $\lambda_w <_\pi \lambda_f$. In the former case the desired result follows immediately. In the latter case, as $\lambda_w <_\pi \lambda_f$, and $\pi{=}\pi_1.\lambda_f.\pi_2$, we have $\lambda_w \in \pi_1$. Moreover, as $\texttt{getVE}(\lambda_{w'}){=}w'$ and thus from $\text{wfp}(\pi)$ we have $\texttt{tid}(w') \neq 0$, i.e. $w' \notin E^0$, from $\text{wffo}(f, w', \pi_1)$ and the uniqueness guarantees of π (given by $\text{wfp}(\pi)$) we know there exist π_a, π_b such that $\pi_1 = \pi_a.\lambda_{w'}.\pi_b$ and $\left\{\lambda' \in \pi_b \mid \exists e' \in ST_x.\ \texttt{getVE}(\lambda') = e'\right\} = \emptyset$. Consequently, as $w \in ST_x$, $\texttt{getVE}(\lambda^p_w){=}w$, $\lambda_w \in \pi_1$ and $\pi_1 = \pi_a.\lambda_{w'}.\pi_b$, we have $\lambda_w \in \pi_a$. That is, as $\pi_1 = \pi_a.\lambda_{w'}.\pi_b$, we have $\lambda_w <_\pi \lambda_{w'}$. This, however, leads to a contradiction as from the assumption of case (ii) we also have $\lambda_{w'} <_\pi \lambda_w$ and $<_\pi$ is a strict total order.

RTS (7)
Follows from the definition of ob and parts (1)–(6).

RTS (8)
Follows from (7) and the fact that $<_\pi$ is a strict total order. □

Lemma 4. *For all π and $G = (E, P, \text{po}, \text{rf}, \text{mo}, \text{pf})$, if $\texttt{getG}(\pi) = G$, then:*

(1) $\forall x \in Loc_{\texttt{nc}} \cup Loc_{\texttt{wt}}, e.\ P(x) = e \Rightarrow e = \max(\text{mo}_x)$

(2) $\forall x \in \textsc{Loc}_{\text{wb}}, e, d.\ P(x) = e \wedge d \in S_x \Rightarrow (d, e) \in \text{mo}^?$
where $S \triangleq NTW_{\text{wb}} \cup dom(\text{pf}; [FL]) \cup dom(\text{pf}; [FO]; \text{po}; [MF \cup SF \cup U])$

Proof. Pick an arbitrary π, $G = (E, P, \text{po}, \text{rf}, \text{mo}, \text{pf})$ such that $\texttt{getG}(\pi) = G$. From the definition of $\texttt{getG}(.)$ we then know $\text{wfp}(\pi)$ holds. We prove each part separately. In what follows we write $\lambda \preccurlyeq_\pi \lambda'$ as a shorthand for $\lambda \prec_\pi \lambda' \vee \lambda{=}\lambda'$.

RTS (1)
Pick arbitrary x, e such that $P(x) = e$ and $x \in \textsc{Loc}_{\text{nc}} \cup \textsc{Loc}_{\text{wt}}$, i.e. $x \notin \textsc{Loc}_{\text{wb}}$. Let us proceed by contradiction and assume that $e \neq \max(\text{mo}_x)$. That is, there exists e' such that $(e, e') \in \text{mo}_x$ and $e, e' \in E \cap ST_x$. From the definition of P we then know that either i) $e \in E \cap ST_x$ and there exist π_1, π_2, λ such that $\pi{=}\pi_1.\lambda.\pi_2$, $\texttt{getPE}(\lambda){=}e$ and $S = \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in E \cap ST_x.\ \texttt{getPE}(\lambda'){=}e'\right\} = \emptyset$; or ii) $e{=}init_x$ and $\neg\exists\lambda, e.\ \lambda \in \pi \wedge \texttt{getPE}(\lambda){=}e \wedge e \in E \cap ST_x$.

In case (i), as $\texttt{getPE}(\lambda){=}e$, $\texttt{loc}(e){=}x$ and $x \notin \textsc{Loc}_{\text{wb}}$, from the definitions of $\texttt{getPE}(.)$ and $\texttt{getVE}(.)$ we also have $\texttt{getVE}(\lambda){=}e$. Moreover, as $(e, e') \in \text{mo}_x$, $\texttt{getVE}(\lambda){=}e$ and $\lambda \in \pi$ (and thus $\texttt{tid}(e) \neq 0$), from the definition of mo we know there exists λ' such that $\texttt{getVE}(\lambda'){=}e'$ and $\lambda \prec_\pi \lambda'$. That is, (since $\pi{=}\pi_1.\lambda.\pi_2$) we have $\lambda' \in \pi_2$. Moreover, as $(e, e') \in \text{mo}_x$ (and thus $\texttt{loc}(e'){=}x$), $e, e' \in ST_x$ and $x \notin \textsc{Loc}_{\text{wb}}$, from the definitions of $\texttt{getPE}(.)$ and $\texttt{getVE}(.)$ we also have $\texttt{getPE}(\lambda'){=}e'$. That is, $\lambda' \in \pi_2$ and $\texttt{getPE}(\lambda'){=}e'$ and thus $\lambda' \in S$. This however leads to a contradiction as $S = \emptyset$.

In case (ii), as $e{=}init_x$ and $(e, e') \in \text{mo}_x$, from the definition of mo we know $e' \in E \setminus E^0$ and thus from the definition of E we know there exist $\lambda' \in \pi$ such that $\texttt{getVE}(\lambda'){=}e'$. Moreover, as $(e, e') \in \text{mo}_x$ (and thus $\texttt{loc}(e'){=}x$), $e' \in ST_x$ and $x \notin \textsc{Loc}_{\text{wb}}$, from the definitions of $\texttt{getPE}(.)$ and $\texttt{getVE}(.)$ we also have $\texttt{getPE}(\lambda'){=}e'$. That is, $\lambda' \in \pi$, $\texttt{getPE}(\lambda'){=}e'$ and $e' \in E \cap ST_x$. This, however, leads to a contradiction as we have $\neg\exists\lambda, e.\ \lambda \in \pi \wedge \texttt{getPE}(\lambda){=}e \wedge e \in E \cap ST_x$.

RTS (2)
Pick arbitrary x, e, d such that $x \in \textsc{Loc}_{\text{wb}}$, $P(x) = e$ and $d \in S_x$ with $S \triangleq NTW_{\text{wb}} \cup dom(\text{pf}; [FL]) \cup dom(\text{pf}; [FO]; \text{po}; [MF \cup SF \cup U])$. Let us proceed by contradiction and assume that $(d, e) \notin \text{mo}^?$. As mo_x is total, we then have $(e, d) \in \text{mo}_x$ and thus $e, d \in E \cap ST_x$. There are then three cases to consider: A) $d \in NTW_{\text{wb}}$; or B) $d \in dom(\text{pf}; [FL])$; or C) $d \in dom(\text{pf}; [FO]; \text{po}; [MF \cup SF \cup U])$.

Case (A): $d \in NTW_{\text{wb}}$
As $P(x) = e$, from the definition of P we then know that either i) $e \in E \cap ST_x$ and there exist π_1, π_2, λ such that $\pi{=}\pi_1.\lambda.\pi_2$, $\texttt{getPE}(\lambda){=}e$ and $S = \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in E \cap ST_x.\ \texttt{getPE}(\lambda'){=}e'\right\} = \emptyset$; or ii) $e{=}init_x$ and $\neg\exists\lambda, e.\ \lambda \in \pi \wedge \texttt{getPE}(\lambda){=}e \wedge e \in E \cap ST_x$.

In case (i), as $\texttt{getPE}(\lambda){=}e$, from wfp() we know there exist $\lambda^v \in \pi$ such that $\texttt{getVE}(\lambda^v) = e$ and $\lambda^v \preccurlyeq_\pi \lambda$. Moreover, as $(e, d) \in \text{mo}_x$, $\texttt{getVE}(\lambda^v){=}e$ and $\lambda^v \in \pi$ (and thus $\texttt{tid}(e) \neq 0$), from the definition of mo we know there exists λ' such that $\texttt{getVE}(\lambda'){=}d$ and $\lambda^v \prec_\pi \lambda'$. Additionally, since $d \in NTW$ and $\texttt{getVE}(\lambda'){=}d$, from the definitions of $\texttt{getPE}(.)$ and $\texttt{getVE}(.)$ we also have $\texttt{getPE}(\lambda'){=}d$. Consequently, as $e, d \in E \cap ST_x$, $x \in \textsc{Loc}_{\text{wb}}$, $\texttt{getVE}(\lambda'){=}d$, $\texttt{getVE}(\lambda^v){=}e$, $\lambda^v \prec_\pi \lambda'$, $\texttt{getPE}(\lambda){=}e$, $\texttt{getPE}(\lambda'){=}d$ and $\lambda' \in \pi$, from $\text{wfp}(\pi)$ (and the uniqueness of its events) we know $alb \prec'_{\lambda}$. That is, since $\pi{=}\pi_1.\lambda.\pi_2$, we have $\lambda' \in \pi_2$. Consequently, as $\lambda' \in \pi_2$, $\texttt{getPE}(\lambda'){=}d$ and $d \in E \cap ST_x$, we have $\lambda' \in S$. This however leads to a contradiction as $S = \emptyset$.

In case (ii), as $e{=}init_x$ and $(e, d) \in \text{mo}_x$, from the definition of mo we know $d \in E \setminus E^0$ and thus from the definition of E we know there exist $\lambda' \in \pi$ such that $\texttt{getVE}(\lambda'){=}d$. Moreover, as $d \in NTW$, from the definitions of $\texttt{getPE}(.)$ and $\texttt{getVE}(.)$ we also have $\texttt{getPE}(\lambda'){=}d$. That is, $\lambda' \in \pi$, $\texttt{getPE}(\lambda'){=}d$ and $d \in E \cap ST_x$. This, however, leads to a contradiction as we have

$\neg\exists\lambda, e.\ \lambda \in \pi \wedge \texttt{getPE}(\lambda)=e \wedge e \in E \cap ST_x$.

Case (B): $d \in dom(\text{pf}; [FL])$
As $(e, d) \in \text{mo}_x$, from the definition of mo and wfp(π) we know $d \in E \setminus E^0$ and that there exists λ_d^v such that $\texttt{getVE}(\lambda_d^v)=d$. On the other hand, as $d \in dom(\text{pf}; [FL])$, we know there exist f, y such that $f \in FL_y \cap E$, $(x, y) \in \text{scl}$ and $(d, f) \in \text{pf}$. As such, given the definition of pf, we know there exist $\lambda_f \in \pi, S$ such that $d \in S$, λ_f=P$\langle f, S\rangle$. Consequently, from wfp(π) we know wffl(f, d, π) holds, and thus (since $dinE \setminus E^0$, i.e. $d \neq init_x$) we know there exist λ_d, π_a, π_b such that π=$\pi_a.\lambda_d.\pi_b$ and $\texttt{getPE}(\lambda_d)=d$. That is, $\lambda_d \in \pi$. Moreover, as $P(x) = e$, from the definition of P we then know that either i) $e \in E \cap ST_x$ and there exist π_1, π_2, λ_e such that π=$\pi_1.\lambda_e.\pi_2$, $\texttt{getPE}(\lambda)_e=e$ and $S = \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in E \cap ST_x.\ \texttt{getPE}(\lambda')=e'\right\} = \emptyset$; or ii) e=$init_x$ and $\neg\exists\lambda, e'.\ \lambda \in \pi \wedge \texttt{getPE}(\lambda)=e' \wedge e' \in E \cap ST_x$.

In case (i), as $\texttt{getPE}(\lambda_e)=e$, from wfp$(\pi)$ we know there exist $\lambda_e^v \in \pi$ such that $\texttt{getVE}(\lambda_e^v) = e$ and $\lambda_e^v \leqslant_\pi \lambda_e$. Moreover, as $(e, d) \in \text{mo}_x$, $\texttt{getVE}(\lambda_e^v)=e$, $\texttt{getVE}(\lambda_d^v)=d$ and $\lambda_e^v \in \pi$ (and thus $\texttt{tid}(e) \neq 0$), from the definition of mo we know $\lambda_e^v <_\pi \lambda_d^v$. Consequently, as $\texttt{getVE}(\lambda_e^v)=e$, $\texttt{getVE}(\lambda_d^v)=d$, $\lambda_e^v <_\pi \lambda_d^v$, $\texttt{getPE}(\lambda)_e=e$, $\texttt{getPE}(\lambda_d)=d$ and $\lambda_d \in \pi$, from wfp() and the uniqueness of labels in π (guaranteed by wfp(π)) we know $\lambda_e <_\pi \lambda_d$. That is, since π=$\pi_1.\lambda_e.\pi_2$, we have $\lambda_d \in \pi_2$. As such, since $d \in E \cap ST_x$, $\texttt{getPE}(\lambda_d)=d$ and $\lambda_d \in \pi_2$, we know $d \in S$, leading to a contradiction since S=$\emptyset$.

In case (ii) we then have $d \in E \cap ST_x$, $\texttt{getPE}(\lambda_d)=d$ and $\lambda_d \in \pi$. This, however, leads to a contradiction as from the assumption of case (ii) we have $\neg\exists\lambda, e'.\ \lambda \in \pi \wedge \texttt{getPE}(\lambda)=e' \wedge e' \in E \cap ST_x$.

Case (C): $d \in dom(\text{pf}; [FO]; \text{po}; [MF \cup SF \cup U])$
As $(e, d) \in \text{mo}_x$, from the definition of mo and wfp(π) we know $d \in E \setminus E^0$ and that there exists λ_d^v such that $\texttt{getVE}(\lambda_d^v)=d$. On the other hand, as $d \in dom(\text{pf}; [FL]; \text{po}; [MF \cup SF \cup U])$, from the definitions of po, pf, E and wfp(π) we know there exist $b, f, y, \lambda_f^v, \lambda_b \in \pi, S$ such that $y \in \text{Loc}_{\text{wb}}$, $f \in FO_y \cap E$, $(x, y) \in \text{scl}$, $b \in (MF \cup SF \cup U) \cap E$, $d \in S$, $\texttt{tid}(f)$=$\texttt{tid}(b)$, λ_f^v=B$\langle f, S\rangle$, λ_b=$\texttt{getVE}(b)$, and $\lambda_f^v <_\pi \lambda_b$. Moreover, from wfp$(\pi)$ and since $d \in S$ and λ_f^v=B$\langle f, S\rangle \in \pi$, we know there exist π_a, π_b such that π=$\pi_a.\lambda_f^v.\pi_b$ and wffo(f, d, π_a). As such, from the the definition of wffo(f, d, π_a), the uniqueness of labels in π (guaranteed by wfp(π)) and since π=$\pi_a.\lambda_f^v.\pi_b$, we know $\lambda_d^v <_\pi \lambda_f^v$. Additionally, as $f \in FO$, $b \in MF \cup SF \cup U$, $\texttt{tid}(f)$=$\texttt{tid}(b)$, λ_f^v=B$\langle f, S\rangle$, λ_b=$\texttt{getVE}(b)$, $\lambda_f^v <_\pi \lambda_b$ and $d \in S$, from wfp(π) we know there exist $\lambda_f \in \pi$ such that $\lambda_f <_\pi \lambda_b$ and λ_f=P$\langle f, d\rangle$. Analogously, as $x, y \in \text{Loc}_{\text{wb}}$, $(x, y) \in \text{scl}$, $d \in ST_x$, $f \in FO_y$, $\texttt{getVE}(\lambda_d^v)=d$, $\texttt{getVE}(\lambda_f^v)=f$, $\lambda_d^v <_\pi \lambda_f^v$, $\lambda_f \in \pi$ and λ_f=P$\langle f, d\rangle$, from wfp(π) we know there exists $\lambda_d \in \pi$ such that $\texttt{getPE}(\lambda_d)=d$ and $\lambda_d <_\pi \lambda_f$.

On the other hand, as $P(x) = e$, from the definition of P we know either i) $e \in E \cap ST_x$ and there exist π_1, π_2, λ_e such that π=$\pi_1.\lambda_e.\pi_2$, $\texttt{getPE}(\lambda)_e=e$ and $S = \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in E \cap ST_x.\ \texttt{getPE}(\lambda')=e'\right\} = \emptyset$; or ii) e=$init_x$ and $\neg\exists\lambda, e'.\ \lambda \in \pi \wedge \texttt{getPE}(\lambda)=e' \wedge e' \in E \cap ST_x$.

In case (i), as $\texttt{getPE}(\lambda_e)=e$, from wfp$(\pi)$ we know there exist $\lambda_e^v \in \pi$ such that $\texttt{getVE}(\lambda_e^v) = e$ and $\lambda_e^v \leqslant_\pi \lambda_e$. Moreover, as $(e, d) \in \text{mo}_x$, $\texttt{getVE}(\lambda_e^v)=e$, $\texttt{getVE}(\lambda_d^v)=d$ and $\lambda_e^v \in \pi$ (and thus $\texttt{tid}(e) \neq 0$), from the definition of mo we know $\lambda_e^v <_\pi \lambda_d^v$. Consequently, as $\texttt{getVE}(\lambda_e^v)=e$, $\texttt{getVE}(\lambda_d^v)=d$, $\lambda_e^v <_\pi \lambda_d^v$, $\texttt{getPE}(\lambda)_e=e$, $\texttt{getPE}(\lambda_d)=d$ and $\lambda_d \in \pi$, from wfp(π) and the uniqueness of labels in π (guaranteed by wfp(π)) we know $\lambda_e <_\pi \lambda_d$. That is, since π=$\pi_1.\lambda_e.\pi_2$, we have $\lambda_d \in \pi_2$. As such, since $d \in E \cap ST_x$, $\texttt{getPE}(\lambda_d)=d$ and $\lambda_d \in \pi_2$, we know $d \in S$, leading to a contradiction since S=$\emptyset$.

In case (ii) we have $d \in E \cap ST_x$, $\texttt{getPE}(\lambda_d)=d$ and $\lambda_d \in \pi$. This, however, leads to a contradiction as from the assumption of case (ii) we have $\neg\exists\lambda, e'.\ \lambda \in \pi \wedge \texttt{getPE}(\lambda)=e' \wedge e' \in E \cap ST_x$. □

Lemma 5. *For all π and $G = (E, P, \text{po}, \text{rf}, \text{mo}, \text{pf})$, if $\texttt{getG}(\pi) = G$, then G is PEx86-consistent.*

Proof. Follows immediately from the definition of PEx86-consistency and Lemmas 2 to 4. □

Theorem 4 (Soundness)**.** *For all* $\mathsf{P}, M, PB, B, \pi, \mathsf{P}'$*, if* $\mathsf{P}, M_0, PB_0, B_0, \epsilon \Rightarrow^* \mathsf{P}', M, PB, B, \pi$*, then there exists an execution* G *such that* G *is PEx86-consistent and* $G.P{=}M$*.*

Proof. Pick arbitrary $\mathsf{P}, M, PB, B, \pi, \mathsf{P}'$ such that $\mathsf{P}, M_0, PB_0, B_0, \epsilon \Rightarrow^* \mathsf{P}', M, PB, B, \pi$. From Lemma 1 we then know $\mathsf{wf}(M, PB, B, \pi)$ and thus $\mathsf{wfp}(\pi)$ holds. As such, from the definition of $\mathtt{getG}(.)$ we know there exists G such that $G{=}\mathtt{getG}(\pi)$. Consequently, from Lemma 5 we know G is PEx86-consistent, as required. Moreover, for each $x \in \mathsf{Loc}$, from $\mathsf{wf}(M, PB, B, \pi)$ we know $M(x){=}\mathtt{pread}(\pi, x)$; analogously, from the definition of $\mathtt{getG}(\pi)$ we know $G.P(x){=}\mathtt{pread}(\pi, x)$. As such, we have $G.P{=}M$, as required. ◻

Completeness of the Event-Annotated Semantics against PEx86 Declarative Semantics

Given an PEx86-consistent execution G=(E, P, po, rf, mo, pf), let ob_t denote an extension of ob to a strict total order on E. Let $e_1, \cdots, e_n$ be an enumeration of $G.E \setminus E^0$ according to ob_t and π^0=$\lambda_1.\cdots.\lambda_n$, where λ_k=$\mathsf{genVL}(e_k, G)$ for $k \in \{1, \cdots, n\}$ and:

$$\mathsf{genVL}(e,G) \triangleq \begin{cases} \mathsf{P}\langle e\rangle & \text{if } e \in \mathit{NTW} \cup W_{\mathsf{nc}} \cup W_{\mathsf{wt}} \\ \mathsf{B}\langle e\rangle & \text{if } e \in W_{\mathsf{wb}} \cup \mathit{SF} \\ \mathsf{B}\langle e,S\rangle & \text{if } e \in \mathit{FO} \text{ with } S=\left\{w \,\middle|\, (w,e) \in \mathsf{pf}\right\} \\ \mathsf{P}\langle e,S\rangle & \text{if } e \in \mathit{FL} \text{ with } S=\left\{w \,\middle|\, (w,e) \in \mathsf{pf}\right\} \\ \mathsf{genL}(e,G) & \text{otherwise} \end{cases}$$

$$\mathsf{genL}(e,G) \triangleq \begin{cases} \mathsf{R}\langle e,w\rangle & \text{if } e \in R \wedge (w,e) \in \mathsf{rf} \\ \mathsf{U}\langle e,w\rangle & \text{if } e \in U \wedge (w,e) \in \mathsf{rf} \\ \mathsf{MF}\langle e\rangle & \text{if } e \in \mathit{MF} \\ \mathsf{SF}\langle e\rangle & \text{if } e \in \mathit{SF} \\ \mathsf{W}\langle e\rangle & \text{if } e \in W \\ \mathsf{NTW}\langle e\rangle & \text{if } e \in \mathit{NTW} \\ \mathsf{FO}\langle e\rangle & \text{if } e \in \mathit{FO} \\ \mathsf{FL}\langle e\rangle & \text{if } e \in \mathit{FL} \end{cases}$$

Let $d_1, \cdots d_m$ denote an enumeration of $(W \cup \mathit{NTW} \cup \mathit{SF} \cup \mathit{FO} \cup \mathit{FL}) \cap (E \setminus E^0)$ that respects po^{-1}. For each $j \in \{1 \cdots m\}$, let $A_j \triangleq \left\{e \,\middle|\, (d_j, e) \in \mathsf{po}\right\}$ and π^j=$\mathsf{addE}(\pi^{j-1}, d_j, A_j)$, where:

$$\mathsf{addE}(\pi,d,A) \triangleq \begin{cases} \mathsf{genL}(d,G).\pi & \text{if } \exists e, \pi'.\ e \in A \wedge \pi=\mathsf{genL}(e,G).\pi' \\ \mathsf{genL}(d,G).\pi & \text{else if } \exists \pi', \lambda.\ \lambda=\mathsf{genVL}(d,G) \wedge \pi=\lambda.\pi' \\ \lambda.\mathsf{addE}(\pi',d,A) & \text{else if } \exists \lambda, \pi'.\ \pi=\lambda.\pi' \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note that for $j \in \{1 \cdots m\}$, π^j is always defined as $\mathsf{genVL}(d_j, G) \in \pi^0$ and thus $\mathsf{genVL}(d_j, G) \in \pi^j$.

Let

$$G.\mathcal{PW} \triangleq \left\{w \in G.W_{\mathsf{wb}} \cup U_{\mathsf{wb}} \setminus E^0 \,\middle|\, \exists x, w'.\ \mathtt{loc}(w)=x \wedge G.P(x)=w' \wedge (w,w') \in G.\mathsf{mo}^?\right\}$$
$$G.\mathcal{PFO} \triangleq \left\{f \in G.\mathit{FO} \,\middle|\, \exists w.\ (w,f) \in G.\mathsf{pf} \wedge w \in \mathcal{PW} \cup \mathit{NTW}\right\}$$

Let $w_1, \cdots w_l$ denote an enumeration of $\mathcal{PW}$ event.Let $\pi_0 \triangleq \pi^m$; for each $j \in \{1 \cdots l\}$, let $\pi_j \triangleq \mathsf{addPW}(\pi_{j-1}, w_j)$, where:

$$\mathsf{addPW}(\pi,w) \triangleq \begin{cases} \pi_1.\lambda.\mathsf{P}\langle w\rangle.\pi_2 & \text{if } \exists \pi_1, \pi_2, \lambda.\ \lambda=\mathsf{genVL}(w,G) \wedge \pi=\pi_1.\lambda.\pi_2 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note that for $j \in \{1 \cdots l\}$, π_j is always defined as $\mathsf{genVL}(w_j, G) \in \pi^0$ and thus $\mathsf{genVL}(w_j, G) \in \pi_j$.

Let $f_1, \cdots, f_k$ denote an enumeration of $\mathcal{PFO}$. Let $S_j \triangleq \left\{w \in \mathit{NTW} \cup \mathcal{PW} \,\middle|\, (w, f_j) \in G.\mathsf{pf}\right\}$, $s_j \triangleq \left|S_j\right|$ and let $[w^j_1 \cdots w^j_{s_j}]$ denote an enumeration of S_j, for each $j \in \{1 \cdots k\}$. Let $\pi'_0 \triangleq \pi_l$; for each $j \in \{1 \cdots k\}$, let $\pi'_j \triangleq \mathsf{addPFO}(\pi'_{j-1}, f_j, [w^j_1 \cdots w^j_{s_j}])$, where:

$$\mathsf{addPFO}(\pi,f,[w_1 \cdots w_n]) \triangleq \begin{cases} \pi_1.\lambda.\mathsf{P}\langle f,w_1\rangle.\cdots.\mathsf{P}\langle f,w_n\rangle.\pi_2 & \text{if } \exists \lambda, \pi_1, \pi_2.\ \lambda=\mathsf{genVL}(f,G) \wedge \pi=\pi_1.\lambda.\pi_2 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note that given the definitions of $\mathcal{PFO}$ and π_l, for all $j \in \{1 \cdots k\}$ we know $\texttt{genVL}(f_j, G) \in \pi_l$. As such, $\texttt{addPFO}(\pi ji, f_j, w_i^j)$ is always defined for all $j \in \{1 \cdots k\}$ and $i \in \{1 \cdots s_j\}$.

Let $\pi \triangleq \pi'_k$. Let us write $\texttt{getPath}(G){=}\pi$ when π is constructed from G as described above.

Proposition 3. *For all PEx86 executions $G{=}(E, P, \text{po}, \text{rf}, \text{mo}, \text{pf})$, and all π, if $\texttt{getPath}(G){=}\pi$, then:*

- $\texttt{nodups}(\pi)$
- $\forall \lambda \in \pi.\ \texttt{tid}(\texttt{getE}(\lambda)) \neq 0$
- $\forall e \in E.\ \texttt{genVL}(e, G) \in \pi$
- $\forall e \in E.\ \texttt{genL}(e, G) \in \pi$
- $\forall e \in FO.\ (\texttt{genPL}(e, G) \subseteq \pi) \vee (\texttt{genPL}(e, G) \cap \pi = \emptyset)$
- $\forall e \in FO.\ \texttt{P}\langle e, -\rangle \in \pi \Rightarrow \texttt{genPL}(e, G) \subseteq \pi$
- $\forall e_1, e_2.\ (e_1, e_2) \in \text{ob} \Rightarrow \texttt{genVL}(e_1, G) <_\pi \texttt{genVL}(e_2, G)$
- $\forall e_1, e_2.\ (e_1, e_2) \in \text{po} \Rightarrow \texttt{genL}(e_1, G) <_\pi \texttt{genL}(e_2, G)$
- $\texttt{PO}(\pi) = \text{po}|_{E \setminus E^0} \subseteq \text{po}$
- $\texttt{PPO}(\pi) = \text{ppo}(\text{po}|_{E \setminus E^0}) \subseteq \text{ppo}(\text{po})$
- $\forall \lambda \in \pi, e.\ \lambda{=}\texttt{genL}(e, G) \Leftrightarrow \texttt{getE}(\lambda){=}e$
- $\forall \lambda \in \pi, e.\ \lambda{=}\texttt{genVL}(e, G) \Leftrightarrow \texttt{getVE}(\lambda){=}e$
- $\forall \lambda \in \pi, e \in ST \cup FL.\ \lambda{=}\texttt{genPL}(e, G) \Leftrightarrow \texttt{getPE}(\lambda){=}e$
- $\forall \lambda \in \pi, e \in FO.\ \lambda \in \texttt{genPL}(e, G) \Leftrightarrow \texttt{getPE}(\lambda){=}e$
- $\forall e \in E.\ \texttt{genL}(e, G) \leqslant_\pi \texttt{genVL}(e, G)$
- $\forall e \in G.(W_{\text{wb}} \cup U_{\text{wb}}), \lambda_p.\ \lambda_p{=}\texttt{genPL}(e, G) \in \pi \Rightarrow \exists \lambda_v.\ \lambda_v{=}\texttt{genVL}(e, G) \wedge \pi{=} - .\lambda_v.\lambda_p.-$
- $\forall e \in G.(ST \setminus (W_{\text{wb}} \cup U_{\text{wb}})).\ \texttt{genPL}(e, G){=}\texttt{genVL}(e, G) \wedge \texttt{genPL}(e, G) \in \pi$
- $\forall e \in G.FO.\ \texttt{genPL}(e, G) \subseteq \pi \Rightarrow \forall \lambda \in \texttt{genPL}(e, G).\ \texttt{genVL}(e, G) \leqslant_\pi \lambda$
- $\forall e \in G.FO.\ \texttt{genPL}(e, G) \subseteq \pi \Rightarrow \exists S.\ \texttt{genVL}(e, G){=}\texttt{B}\langle e, S\rangle \wedge \forall w \in S.\ \texttt{genPL}(w, G) <_\pi \texttt{P}\langle e, w\rangle$

where $\texttt{genPL}(., G) : (G.(ST \cup FL) \rightarrow ALabels) \cup (G.FO \rightarrow \mathcal{P}(ALabels))$ *is defined as:*

$$\texttt{genPL}(e, G) \triangleq \begin{cases} \texttt{P}\langle e\rangle & \text{if } e \in W_{\text{wb}} \cup U_{\text{wb}} \\ \left\{ \texttt{P}\langle e, w\rangle \;\middle|\; \begin{array}{l} (w, e) \in G.\text{pf} \wedge \\ (w \in NTW \vee (w, G.P(\texttt{loc}(w))) \in G.\text{mo}^?) \end{array} \right\} & \text{if } e \in FO \\ \texttt{genVL}(e, G) & \text{otherwise} \end{cases}$$

Lemma 6. *For all PEx86-consistent executions G and all π, if $\texttt{getPath}(G){=}\pi$, then $\texttt{wfp}(\pi)$ holds.*

Proof. Pick an arbitrary PEx86-consistent execution $G{=}(E, P, \text{po}, \text{rf}, \text{mo}, \text{pf})$ and π such that $\texttt{getPath}(G){=}\pi$. We are then required to show that for all $\lambda, \pi_1, \pi_2, e, r, u, e_1, e_2, \lambda_1, \lambda_2, x, y, S$:

$$\texttt{nodups}(\pi) \wedge \forall \lambda \in \pi.\ \texttt{tid}(\texttt{getE}(\lambda)) \neq 0 \tag{1}$$
$$\pi{=}\pi_1.\texttt{R}\langle r, e\rangle.\pi_2 \vee \pi{=}\pi_1.\texttt{U}\langle u, e\rangle.\pi_2 \Rightarrow \texttt{wfrd}(r, e, \pi_1) \tag{2}$$
$$\pi{=}\pi_1.\texttt{P}\langle e, S\rangle.\pi_2 \wedge e \in FL \Rightarrow \forall w \in S.\ \texttt{wffl}(e, w, \pi_1) \tag{3}$$
$$\pi{=}\pi_1.\texttt{B}\langle e, S\rangle.\pi_2 \wedge e \in FO \Rightarrow \forall w \in S.\ \texttt{wffo}(e, w, \pi_1) \tag{4}$$
$$\pi{=}\pi_1.\texttt{P}\langle e, w\rangle.\pi_2 \wedge e \in FO \Rightarrow \texttt{wfpfo}(e, w, \pi_1) \tag{5}$$
$$\lambda \in \pi \wedge \texttt{getVE}(\lambda) = e \Rightarrow \exists! \lambda'.\ \lambda' \leqslant_\pi \lambda \wedge \texttt{getE}(\lambda') = e \tag{6}$$
$$\lambda \in \pi \wedge \texttt{getPE}(\lambda) = e \Rightarrow \exists! \lambda'.\ \lambda' \leqslant_\pi \lambda \wedge \texttt{getVE}(\lambda') = e \tag{7}$$
$$(e_1, e_2) \in \texttt{PPO}(\pi) \wedge \lambda_2 \in \pi \wedge \texttt{getVE}(\lambda_2){=}e_2 \Rightarrow \exists! \lambda_1.\ \lambda_1 <_\pi \lambda_2 \wedge \texttt{getVE}(\lambda_1){=}e_1 \tag{8}$$
$$\lambda \in \pi \wedge \lambda{=}\texttt{P}\langle e, w\rangle \wedge e \in FO \Rightarrow \exists S.\ w \in S \wedge \texttt{B}\langle e, S\rangle <_\pi \lambda \tag{9}$$

$$\left(\begin{array}{l} e_1, e_2 \in ST_x \wedge \texttt{getVE}(\lambda_1)=e_1 \wedge \texttt{getVE}(\lambda_2)=e_2 \wedge \lambda_1 <_\pi \lambda_2 \wedge \lambda \in \pi \wedge \texttt{getPE}(\lambda) = e_2 \\ \qquad\qquad \Rightarrow \exists \lambda'.\ \texttt{getPE}(\lambda') = e_1 \wedge \lambda' <_\pi \lambda \end{array}\right) \tag{10}$$

$$\left(\begin{array}{l} x, y \in \mathsf{Loc}_{\mathsf{wb}} \wedge (x, y) \in \mathsf{scl} \wedge e_1 \in ST_x \wedge e_2 \in FL_y \\ \wedge\ \texttt{getVE}(\lambda_1)=e_1 \wedge \texttt{getVE}(\lambda_2)=e_2 \wedge \lambda_1 <_\pi \lambda_2 \\ \qquad\qquad \Rightarrow \exists \lambda'.\ \texttt{getPE}(\lambda') = e_1 \wedge \lambda' <_\pi \lambda_2 \end{array}\right) \tag{11}$$

$$\left(\begin{array}{l} x, y \in \mathsf{Loc}_{\mathsf{wb}} \wedge (x, y) \in \mathsf{scl} \wedge e_1, e \in ST_x \wedge e_2 \in FO_y \\ \wedge\ \texttt{getVE}(\lambda_1)=e_1 \wedge \texttt{getVE}(\lambda_2)=e_2 \wedge \lambda_1 <_\pi \lambda_2 \wedge \lambda=\mathsf{P}\langle e_2, e\rangle \wedge \lambda \in \pi \\ \qquad\qquad \Rightarrow \exists \lambda'.\ \texttt{getPE}(\lambda') = e_1 \wedge \lambda' <_\pi \lambda \end{array}\right) \tag{12}$$

$$\left(\begin{array}{l} x, y \in \mathsf{Loc}_{\mathsf{wb}} \wedge (x, y) \in \mathsf{scl} \wedge e_1 \in FO_y \wedge e_2 \in ST_x \\ \wedge\ \texttt{getVE}(\lambda_1)=e_1 \wedge \texttt{getVE}(\lambda_2)=e_2 \wedge \lambda_1 <_\pi \lambda_2 \wedge e_2=\texttt{getPE}(\lambda) \wedge \lambda \in \pi \\ \qquad\qquad \Rightarrow \exists e \in ST_x.\ \mathsf{P}\langle e_1, e\rangle <_\pi \lambda \end{array}\right) \tag{13}$$

$$\left(\begin{array}{l} e_1, e_2 \in FO \wedge (\texttt{loc}(e_1), \texttt{loc}(e_2)) \in \mathsf{scl} \wedge \texttt{getVE}(\lambda_1)=e_1 \wedge \texttt{getVE}(\lambda_2)=e_2 \\ \wedge\ \lambda_1 <_\pi \lambda_2 \wedge \mathsf{P}\langle e_2, e\rangle \in \pi \\ \qquad\qquad \Rightarrow \exists e' \in ST_{\texttt{loc}(e)}.\ \mathsf{P}\langle e_1, e'\rangle <_\pi \mathsf{P}\langle e_2, e\rangle \end{array}\right) \tag{14}$$

$$\left(\begin{array}{l} e_1 \in FO \wedge e_2 \in FL \wedge (\texttt{loc}(e_1), \texttt{loc}(e_2)) \in \mathsf{scl} \wedge \lambda_1=\mathsf{B}\langle e_1, S\rangle \wedge \texttt{getVE}(\lambda_2)=e_2 \wedge \lambda_1 <_\pi \lambda_2 \\ \qquad\qquad \Rightarrow \forall e' \in S.\ \mathsf{P}\langle e_1, e'\rangle <_\pi \lambda_2 \end{array}\right) \tag{15}$$

$$\left(\begin{array}{l} e_1 \in FO \wedge e_2 \in MF \cup SF \cup U \wedge \texttt{tid}(e_1)=\texttt{tid}(e_2) \wedge \mathsf{B}\langle e_1, S\rangle <_\pi \lambda_2 \wedge \texttt{getVE}(\lambda_2)=e_2 \\ \qquad\qquad \Rightarrow \forall w \in S.\ \mathsf{P}\langle e_1, w\rangle <_\pi \lambda_2. \end{array}\right) \tag{16}$$

The proofs of parts (1), (6), (7) and (9) follow from Prop. 3.

RTS (2)
Pick arbitrary $\pi_a, \pi_b, r, e, \lambda_r$ such that $\pi{=}\pi_a.\lambda_r.\pi_b$, $\lambda_r{=}\mathsf{R}\langle r, e\rangle \vee \lambda_r{=}\mathsf{U}\langle r, e\rangle$. That is, we have $\texttt{getE}(\lambda_r){=}\texttt{getVE}(\lambda_r){=}$ and (from Prop. 3) $\texttt{genL}(r, G){=}\texttt{genVL}(r, G){=}\lambda_r$. From the construction of π we then know $(e, r) \in \mathsf{rf}$ and there exists x such that $\texttt{loc}(e){=}\texttt{loc}(r){=}x$ and $e \in ST_x$. There are two cases to consider: 1) $e \in E \setminus E^0$; 2) $e \in E^0$.

Case (1)
As G is PEx86-consistent, we know that $(e, r) \in \mathsf{rf}_i \cup \mathsf{rf}_e \subseteq \mathsf{po} \cup \mathsf{ob}$. There are thus two sub-cases to consider: i) $(e, r) \in \mathsf{ob}$; or ii) $(e, r) \in \mathsf{po} \setminus \mathsf{ob}$.

In case (i) from the construction of π (Prop. 3) we know there exists π_1, π_2 such that $\pi_a = \pi_1.\lambda.\pi_2$ and $\lambda{=}\texttt{genVL}(e, G)$ and thus (from Prop. 3) $\texttt{getVE}(\lambda){=}e$. Let us assume there exists $e' \in ST_x, \lambda'$ such that $\texttt{getVE}(\lambda'){=}e'$ and $\lambda' \in \pi_2$. From Prop. 3 we then know $\texttt{genVL}(e', G){=}\lambda'$. That is, since $\pi{=}\pi_a.\lambda_r.\pi_b$, $\pi_a = \pi_1.\lambda.\pi_2$, $\lambda{=}\texttt{genVL}(e, G)$ and $\lambda'{=}\texttt{genVL}(e', G) \in \pi_2$, we know $\texttt{genVL}(e, G) <_\pi \texttt{genVL}(e', G)$. Consequently, as $\mathsf{mo} \subseteq \mathsf{ob}$, mo is total on ST_x and $\texttt{genVL}(e, G) <_\pi \texttt{genVL}(e', G)$, from Prop. 3 we know $(e, e') \in \mathsf{mo}$. As such since we have $(e, r) \in \mathsf{rf}$, we also have $(r, e') \in \mathsf{rb} \subseteq \mathsf{rb}_i \cup \mathsf{rb}_e$ and thus (from the consistency of G) $(r, e') \in \mathsf{po} \cup \mathsf{ob}$. In the former case, if $(r, e') \in \mathsf{po}$ then $\lambda_r <_\pi \texttt{genL}(e', G)$ and thus from Prop. 3 $\texttt{genL}(r, G) <_\pi \texttt{genL}(e', G) <_\pi \texttt{genVL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda' \in \pi_b$ and $\lambda' \notin \pi_2$, contradicting our assumption that $\lambda' \in \pi_2$. Similarly, in the latter case if $(r, e') \in \mathsf{ob}$ then $\lambda_r{=}\texttt{genVL}(r, G) <_\pi \texttt{genVL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda' \in \pi_b$ and $\lambda' \notin \pi_2$, contradicting our assumption that $\lambda' \in \pi_2$. We can thus conclude that $\left\{\lambda' \in \pi_2 \,\middle|\, \texttt{getVE}(\lambda') \in ST_x\right\} = \emptyset$, as required.

Similarly, let us assume there exists $e' \in ST_x, \lambda'$ such that $\texttt{getE}(\lambda'){=}e'$, $\texttt{tid}(r){=}\texttt{tid}(e')$, $\lambda' \in \pi_a$ and $\forall \lambda \in \pi_a.\ \texttt{getVE}(\lambda'') \neq e''$. From Prop. 3 we then know $\texttt{genL}(e', G){=}\lambda'$. From Prop. 3 we know there exists $\lambda'_v \in \pi_b$ such that $\texttt{genVL}(e', G){=}\lambda'_v$. That is, since $\pi{=}\pi_a.\lambda_r.\pi_b$, $\pi_a = \pi_1.\lambda.\pi_2$,

λ=$\texttt{genVL}(e, G)$ and λ'_v=$\texttt{genVL}(e', G) \in \pi_b$, we know $\texttt{genVL}(e, G) \prec_\pi \texttt{genVL}(e', G)$. Consequently, as mo ⊆ ob, mo is total on ST_x and $\texttt{genVL}(e, G) \prec_\pi \texttt{genVL}(e', G)$, from Prop. 3 we know $(e, e') \in$ mo. As such since we have $(e, r) \in$ rf, we also have $(r, e') \in$ rb. Moreover, as $\texttt{tid}(r)$=$\texttt{tid}(e')$ we have $(r, e') \in \text{rb}_i$ and thus (from the consistency of G) $(r, e') \in$ po. As such, from Prop. 3 $\texttt{genL}(r, G) \prec_\pi \texttt{genL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda' \in \pi_b$ and $\lambda' \notin \pi_a$, contradicting our assumption that $\lambda' \in \pi_a$. We can thus conclude the following as required:

$$\left\{\lambda' \in \pi_a \,\middle|\, \exists e' \in ST_x.\ \texttt{getE}(\lambda')=e' \wedge \texttt{tid}(e')=\texttt{tid}(r) \wedge \forall \lambda'' \in \pi_a.\ \texttt{getVE}(\lambda'') \neq e'\right\} = \emptyset$$

In case (ii) from the construction of π (Prop. 3) we know there exists π_1, π_2 such that $\pi_a = \pi_1.\lambda.\pi_2$, $\texttt{tid}(r)$=$\texttt{tid}(e)$ and λ=$\texttt{genL}(e, G)$ and thus (from Prop. 3) $\texttt{getE}(\lambda)$=e. We then know that either $\texttt{genVL}(e, G) \in \pi_a$ or $\texttt{genVL}(e, G) \notin \pi_a$. In the former case the desired result follows from the proof of case (i). In the latter case we then have $\forall \lambda' \in \pi_a.\ \texttt{getVE}(\lambda') \neq e$, as required. Let us let us assume there exists $\lambda' \in \pi_2, e' \in ST_x$ such that $\texttt{tid}(e') = \texttt{tid}(r)$ and $\texttt{getVE}(\lambda')$=e'. From Prop. 3 we know $\texttt{genL}(e', G)$=λ'. As λ=$\texttt{genL}(e, G)$, $\pi_a = \pi_1.\lambda.\pi_2$, $\pi_a = \pi_1.\lambda.\pi_2$, $\lambda' \in \pi_2$, $\texttt{genL}(e', G)$=λ' and thus $genLeG \prec_\pi \texttt{genL}(e', G)$. Moreover, as $\texttt{tid}(e)$=$\texttt{tid}(r)$=$\texttt{tid}(e')$, we know that either $(e, e') \in$ po or $(e', e) \in$ po. As such, since $genLeG \prec_\pi \texttt{genL}(e', G)$, from Prop. 3 we have $(e, e') \in$ po and thus since $e, e' \in ST_x$ and G is consistent, we also have $(e, e') \in$ mo. Additionally, since $(e, r) \in$ rf and $(e, e') \in$ mo, we have $(r, e') \in$ rb; since $\texttt{tid}(r)$=$\texttt{tid}(e')$, we have $(r, e') \in \text{rb}_i$ and thus since G is PEx86-consistent we have $(r, e') \in$ po. As such, from Prop. 3 $\texttt{genL}(r, G) \prec_\pi \texttt{genL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda' \in \pi_b$ and $\lambda' \notin \pi_2$, contradicting our assumption that $\lambda' \in \pi_2$. We can thus conclude that $\left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \texttt{getE}(\lambda')=e' \wedge \texttt{tid}(e')=\texttt{tid}(r)\right\} = \emptyset$, as required.

Case (2)
In case (2), as G is PEx86-consistent, we know $e = \texttt{init}(x)$. Let us now assume there exists $\lambda \in \pi_a, e' \in ST_x$ such that either i) $\texttt{getVE}(\lambda)$=e' and thus (from Prop. 3) $\texttt{genVL}(e', G)$=λ or ii) $(\texttt{getE}(\lambda)=e' \wedge \texttt{tid}(e')=\texttt{tid}(r))$ and thus (from Prop. 3) $\texttt{genL}(e', G)$=λ.

In case (i), since G is PEx86-consistent, we know $(e, e') \in$ mo and thus since $(e, r) \in$ rf we also have $(r, e') \in \text{rb} \subseteq \text{rb}_i \cup \text{rb}_e$ and thus (since G is PEx86-consistent) $(r, e') \in \text{po} \cup \text{ob}$. In the former case, if $(r, e') \in$ po then $\lambda_r \prec_\pi \texttt{genL}(e', G)$ and thus from Prop. 3 $\texttt{genL}(r, G) \prec_\pi \texttt{genL}(e', G) \prec_\pi \texttt{genVL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda \in \pi_b$ and $\lambda \notin \pi_a$, contradicting our assumption that $\lambda \in \pi_a$. Similarly, in the latter case if $(r, e') \in$ ob then λ_r=$\texttt{genVL}(r, G) \prec_\pi \texttt{genVL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda \in \pi_b$ and $\lambda \notin \pi_a$, contradicting our assumption that $\lambda \in \pi_a$. We can thus conclude that $\left\{\lambda \in \pi_a \,\middle|\, \exists e' \in ST_x.\ \texttt{getVE}(\lambda)=e' \vee (\texttt{getE}(\lambda)=e' \wedge \texttt{tid}(e')=\texttt{tid}(r))\right\} = \emptyset$, as required.

Similarly, in case (ii), since G is PEx86-consistent, we know $(e, e') \in$ mo and thus since $(e, r) \in$ rf we also have $(r, e') \in$ rb. Moreover, as $\texttt{tid}(r)$=$\texttt{tid}(e')$ we have $(r, e') \in \text{rb}_i$ and thus (since G is PEx86-consistent) $(r, e') \in$ po. Consequently, from Prop. 3 we have $\lambda_r \prec_\pi \texttt{genL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda \in \pi_b$ and $\lambda \notin \pi_a$, contradicting our assumption that $\lambda \in \pi_a$. We can thus conclude that $\left\{\lambda \in \pi_a \,\middle|\, \exists e' \in ST_x.\ \texttt{getVE}(\lambda)=e' \vee (\texttt{getE}(\lambda)=e' \wedge \texttt{tid}(e')=\texttt{tid}(r))\right\}$ $= \emptyset$, as required.

RTS (3)
Pick arbitrary $\pi_a, \pi_b, f, S, e, \lambda_f$ such that π=$\pi_a.\lambda_f.\pi_b$, λ_f=$\text{P}\langle f, S\rangle$ and $e \in S$. That is, we have $\texttt{getVE}(\lambda_f)$=$f$ and (from Prop. 3) $\texttt{genVL}(f, G)$=λ_f. From the construction of π we then know $(e, f) \in$ pf and there exists $x, y \in \text{Loc}_{\text{wb}}$ such that $\texttt{loc}(e)$=x, $\texttt{loc}(f)$=y, $(x, y) \in$ scl, $f \in FL_y$ and $e \in ST_x$. There are two cases to consider: 1) $e \in E \setminus E^0$; 2) $e \in E^0$.

Case (1)
As G is PEx86-consistent, we know that $(e, f) \in \text{pf} \subseteq \text{ob}$. From the construction of π (Prop. 3) we know there exists π_1, π_2 such that $\pi_a = \pi_1.\lambda.\pi_2$ and λ=$\texttt{genVL}(e, G)$ and thus (from Prop. 3) $\texttt{getVE}(\lambda)$=e. Let us assume there exists $e' \in ST_x, \lambda'$ such that $\texttt{getPE}(\lambda')$=e' and $\lambda' \in \pi_2$. From Prop. 3 we then know $\texttt{genPL}(e', G)$=λ', and that there exists λ'_v=$\texttt{genVL}(e', G)$ such that either λ'_v=λ' or π_2=$-.\lambda'_v.\lambda'.-$. That is, since $\pi_a = \pi_1.\lambda.\pi_2$, we have $\lambda <_\pi \lambda'_v$, and thus $\texttt{genVL}(e, G) <_\pi \texttt{genVL}(e', G)$, and $\lambda'_v \in \pi_2$. Consequently, as $\text{mo} \subseteq \text{ob}$, mo is total on ST_x and $\texttt{genVL}(e, G) <_\pi \texttt{genVL}(e', G)$, from Prop. 3 we know $(e, e') \in \text{mo}$. As such since we have $(e, f) \in \text{pf}$, we also have $(f, e') \in \text{pb}$ and thus (from the consistency of G) $(f, e') \in \text{ob}$. As such, from Prop. 3 we know λ_f=$\texttt{genVL}(f, G) <_\pi \texttt{genVL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda'_v \in \pi_b$ and $\lambda'_v \notin \pi_2$, contradicting our result earlier that $\lambda' \in \pi_2$. We can thus conclude that $\left\{\lambda' \in \pi_2 \mid \exists e' \in ST_x.\ \texttt{getPE}(\lambda')\text{=}e'\right\} = \emptyset$, as required.

Case (2)
As G is PEx86-consistent, we know $e = \texttt{init}(x)$. Let us now assume there exists $\lambda \in \pi_a, e' \in ST_x$ such that $\texttt{getPE}(\lambda)$=e' and thus (from Prop. 3) we know $\texttt{genPL}(e', G)$=λ, and that there exists λ_v=$\texttt{genVL}(e', G)$ such that either λ_v=λ or π_a=$-.\lambda_v.\lambda.-$, and thus $\lambda_v \in \pi_a$. As G is PEx86-consistent, we know $(e, e') \in \text{mo}$ and thus since $(e, f) \in \text{pf}$ we also have $(f, e') \in \text{pb}$ and thus (since G is PEx86-consistent) $(f, e') \in \text{ob}$. As such, from Prop. 3 we know λ_f=$\texttt{genVL}(f, G) <_\pi \texttt{genVL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda_v \in \pi_b$ and $\lambda_v \notin \pi_a$, contradicting our result earlier that $\lambda_v \in \pi_a$. We can thus conclude that $\left\{\lambda \in \pi_a \mid \exists e' \in ST_x.\ \texttt{getPE}(\lambda)\text{=}e'\right\} = \emptyset$, as required.

RTS (4)
Pick arbitrary $\pi_a, \pi_b, f, S, e, \lambda_f$ such that π=$\pi_a.\lambda_f.\pi_b$, λ_f=$\text{B}\langle f, S\rangle$ and $e \in S$. That is, we have $\texttt{getVE}(\lambda_f)$=$f$ and (from Prop. 3) $\texttt{genVL}(f, G)$=λ_f. From the construction of π we then know $(e, f) \in \text{pf}$ and there exists $x, y \in \text{Loc}_{\text{wb}}$ such that $\texttt{loc}(e)$=x, $\texttt{loc}(f)$=y, $(x, y) \in \text{scl}$, $f \in FO_y$ and $e \in ST_x$. There are two cases to consider: 1) $e \in E \setminus E^0$; 2) $e \in E^0$.

Case (1)
As G is PEx86-consistent, we know that $(e, f) \in \text{pf} \subseteq \text{ob}$. From the construction of π (Prop. 3) we know there exists π_1, π_2 such that $\pi_a = \pi_1.\lambda.\pi_2$ and λ=$\texttt{genVL}(e, G)$ and thus (from Prop. 3) $\texttt{getVE}(\lambda)$=e. Let us assume there exists $e' \in ST_x, \lambda'$ such that $\texttt{getVE}(\lambda')$=e' and $\lambda' \in \pi_2$. From Prop. 3 we then know $\texttt{genVL}(e', G)$=λ'. That is, since $\pi_a = \pi_1.\lambda.\pi_2$, we have $\lambda <_\pi \lambda'$, and thus $\texttt{genVL}(e, G) <_\pi \texttt{genVL}(e', G)$. Consequently, as $\text{mo} \subseteq \text{ob}$, mo is total on ST_x and $\texttt{genVL}(e, G) <_\pi \texttt{genVL}(e', G)$, from Prop. 3 we know $(e, e') \in \text{mo}$. As such since we have $(e, f) \in \text{pf}$, we also have $(f, e') \in \text{pb}$ and thus (from the consistency of G) $(f, e') \in \text{ob}$. As such, from Prop. 3 we know λ_f=$\texttt{genVL}(f, G) <_\pi \texttt{genVL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda' \in \pi_b$ and $\lambda' \notin \pi_2$, contradicting our assumption that $\lambda' \in \pi_2$. We can thus conclude that $\left\{\lambda' \in \pi_2 \mid \exists e' \in ST_x.\ \texttt{getVE}(\lambda')\text{=}e'\right\} = \emptyset$, as required.

Case (2)
As G is PEx86-consistent, we know $e = \texttt{init}(x)$. Let us now assume there exists $\lambda \in \pi_a, e' \in ST_x$ such that $\texttt{getVE}(\lambda)$=e' and thus (from Prop. 3) we know $\texttt{genVL}(e', G)$=λ. As G is PEx86-consistent, we know $(e, e') \in \text{mo}$ and thus since $(e, f) \in \text{pf}$ we also have $(f, e') \in \text{pb}$ and thus (since G is PEx86-consistent) $(f, e') \in \text{ob}$. As such, from Prop. 3 we know λ_f=$\texttt{genVL}(f, G) <_\pi \texttt{genVL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda \in \pi_b$ and $\lambda \notin \pi_a$, contradicting our

assumption that $\lambda \in \pi_a$. We can thus conclude that $\left\{\lambda \in \pi_a \,\middle|\, \exists e' \in ST_x.\ \texttt{getVE}(\lambda){=}e'\right\} = \emptyset$, as required.

RTS (3)
Pick arbitrary $\pi_c, \pi_b, f, e, \lambda_f$ such that $\pi{=}\pi_c.\lambda_f.\pi_b$ and $\lambda_f{=}\mathsf{P}\langle f, e\rangle$. That is, we have $\texttt{getPE}(\lambda_f){=}f$ and (from Prop. 3) $\lambda_f \in \texttt{genPL}(f, G)$. From the construction of π we then know $(e, f) \in \mathsf{pf}$ and there exist $x, y \in \mathsf{Loc}_{\mathsf{wb}}, \lambda^v_f$ such that $\texttt{loc}(e){=}x$, $\texttt{loc}(f){=}y$, $(x, y) \in \mathsf{scl}$, $f \in FO_y$, $e \in NTW_x \cup \mathcal{PW}_x$, $\texttt{genVL}(f, G){=}\lambda^v_f$ and thus (from Prop. 3) $\texttt{getVE}(\lambda^v_f){=}f$, and λ_f appears immediately after λ^v_f: there exists π_a such that $\pi{=}\pi_a.\lambda^v_f.\lambda_f.\pi_b$. There are two cases to consider: 1) $e \in E \setminus E^0$; 2) $e \in E^0$.

Case (1)
As G is PEx86-consistent, we know $(e, f) \in \mathsf{pf} \subseteq \mathsf{ob}$. From the construction of π (Prop. 3) we know there exists π_1, π_2 such that $\pi_a = -.\lambda.-$ and $\lambda{=}\texttt{genVL}(e, G)$ and thus (from Prop. 3) $\texttt{getVE}(\lambda){=}e$. Moreover, as $e \in NTW_x \cup \mathcal{PW}_x$, we know that either (when $e \in NTW_x$) $\texttt{getPE}(\lambda){=}e$ and $\lambda{=}\texttt{genPL}(e, G)$, or (when $e \in \mathcal{PW}$) there exists λ^p such that $\texttt{getPE}(\lambda){=}e$, $\lambda{=}\texttt{genPL}(e, G)$ and λ^p appears immediately after λ in π: $\pi_a = -.\lambda.\lambda^p.-$. As such, in both cases we know there exist π_1, π_2, λ^p such that $\pi_a = \pi_1.\lambda^p.\pi_2$, $\lambda \leqslant_\pi \lambda^p$, $\texttt{getPE}(\lambda^p){=}e$ and $\lambda^p{=}\texttt{genPL}(e, G)$.

Let us assume there exists $e' \in ST_x, \lambda'$ such that $\texttt{getPE}(\lambda'){=}e'$ and $\lambda' \in \pi_2$. From Prop. 3 we then know $\texttt{genPL}(e', G){=}\lambda'$, and that there exists $\lambda'_v{=}\texttt{genVL}(e', G)$ such that either $\lambda'_v{=}\lambda'$ or $\pi_2{=}-.\lambda'_v.\lambda'.-$. That is, since $\pi_a = \pi_1.\lambda^p.\pi_2$ and $\lambda \leqslant_\pi \lambda^p$, we have $\lambda <_\pi \lambda'_v$, and thus $\texttt{genVL}(e, G) <_\pi \texttt{genVL}(e', G)$, and $\lambda'_v \in \pi_2$. Consequently, as $\mathsf{mo} \subseteq \mathsf{ob}$, mo is total on ST_x and $\texttt{genVL}(e, G) <_\pi \texttt{genVL}(e', G)$, from Prop. 3 we know $(e, e') \in \mathsf{mo}$. As such since we have $(e, f) \in \mathsf{pf}$, we also have $(f, e') \in \mathsf{pb}$ and thus (from the consistency of G) $(f, e') \in \mathsf{ob}$. Therefore, from Prop. 3 we know $\lambda^v_f{=}\texttt{genVL}(f, G) <_\pi \texttt{genVL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda'_v \in \pi_b$ and $\lambda'_v \notin \pi_2$, contradicting our result earlier that $\lambda' \in \pi_2$. We can thus conclude that $\left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \texttt{getPE}(\lambda'){=}e'\right\} = \emptyset$, as required.

Case (2)
As G is PEx86-consistent, we know $e = \texttt{init}(x)$ and thus $e \in \mathcal{PW}$. Let us now assume there exists $\lambda \in \pi_a, e' \in ST_x$ such that $\texttt{getPE}(\lambda){=}e'$ and thus (from Prop. 3) we know $\texttt{genPL}(e', G){=}\lambda$, and that there exists $\lambda_v{=}\texttt{genVL}(e', G)$ such that either $\lambda_v{=}\lambda$ or $\pi_a{=}-.\lambda_v.\lambda.-$, and thus $\lambda_v \in \pi_a$. As G is PEx86-consistent, we know $(e, e') \in \mathsf{mo}$ and thus since $(e, f) \in \mathsf{pf}$ we also have $(f, e') \in \mathsf{pb}$ and thus (since G is PEx86-consistent) $(f, e') \in \mathsf{ob}$. As such, from Prop. 3 we know $\lambda^v_f{=}\texttt{genVL}(f, G) <_\pi \texttt{genVL}(e', G)$; i.e. (from the uniqueness of labels in π given by Prop. 3) $\lambda_v \in \pi_b$ and $\lambda_v \notin \pi_a$, contradicting our result earlier that $\lambda_v \in \pi_a$. We can thus conclude that $\left\{\lambda \in \pi_a \,\middle|\, \exists e' \in ST_x.\ \texttt{getPE}(\lambda){=}e'\right\} = \emptyset$, as required.

RTS (8)
Pick arbitrary e_1, e_2, λ_2 such that $(e_1, e_2) \in \mathsf{PPO}(\pi)$, $\lambda_2 \in \pi$ and $\texttt{getVE}(\lambda_2){=}e_2$, and thus from Prop. 3 we have $\texttt{genVL}(e_2, G){=}\lambda_2$. From Prop. 3 we then know $(e_1, e_2) \in \mathsf{ppo}(\mathsf{po})$ and thus since G is consistent, we know $(e_1, e_2) \in \mathsf{ob}$. As such, from Prop. 3 we know $\texttt{genVL}(e_1, G) <_\pi \texttt{genVL}(e_2,)$. Consequently, as $\texttt{genVL}(e_2, G){=}\lambda_2$, from Prop. 3 and the uniqueness of its labels we know there exists a unique λ_1 such that $\lambda_1 <_\pi \lambda_2$, $\texttt{genVL}(e_1, G){=}\lambda_1$ and thus $\texttt{getVE}(\lambda_1){=}e_1$, as required.

RTS (10)
Pick arbitrary $x, e_1, e_2 \in ST_x, \lambda_1, \lambda_2, \lambda$ such that $\texttt{getVE}(\lambda_1){=}e_1$, $\texttt{getVE}(\lambda_2){=}e_2$, $\lambda_1 <_\pi \lambda_2$, $\lambda \in \pi$ and $\texttt{getPE}(\lambda) = e_2$. From Prop. 3 we know $\texttt{genVL}(e_1, G){=}\lambda_1$, $\texttt{genVL}(e_2, G){=}\lambda_2$, $\texttt{genVL}(e_1, G) <_\pi$

$\texttt{genVL}(e_2,)$, $\texttt{genPL}(e_2, G)=\lambda$ and $e_1, e_2 \notin E^0$. Moreover, from Prop. 3 we know $\lambda_2 \preccurlyeq_\pi \lambda$. There are now two cases to consider: i) $e_1 \in ST \setminus (W_{\text{wb}} \cup U_{\text{wb}})$; or ii) $e_1 \in W_{\text{wb}} \cup U_{\text{wb}}$.

In case (i) we then have $\texttt{genPL}(e_1, G)=\texttt{genVL}(e_1, G)=\lambda_1$ and thus $\texttt{getPE}(\lambda_1)=e_1$. Consequently, as we have $\lambda_1 \prec_\pi \lambda_2$ and $\lambda_2 \preccurlyeq_\pi \lambda$, we have $\lambda_1 \prec_\pi \lambda$, as required.

In case (ii), as $\text{mo} \subseteq \text{ob}$, mo is total on ST_x and $\texttt{genVL}(e_1, G) \prec_\pi \texttt{genVL}(e_2, G)$, from Prop. 3 we know $(e_1, e_2) \in \text{mo}$. As $e_1, e_2 \in ST_x$ and $e_1 \in W_{\text{wb}} \cup U_{\text{wb}}$, we know $x \in \text{Loc}_{\text{wb}}$. Pick w such that $P(x)=w$. As $e_2 \in ST_x$ and $x \in \text{Loc}_{\text{wb}}$, we know either: a) $e_2 \in NTW_{\text{wb}}$ and thus since G is consistent $(e_2, w) \in \text{mo}^?$ (from WEAK-PERSIST); or b) $e_2 \in W_{\text{wb}} \cup U_{\text{wb}}$ and thus from the construction of π and since $\texttt{genPL}(e_2, G)=\lambda \in \pi$ and $e_2 \notin E^0$, we know $e_2 \in \mathcal{PW}$; that is, from the definition of $\mathcal{PW}$ we have $(e_2, w) \in \text{mo}^?$. As in both cases (a) and (b) we have $(e_2, w) \in \text{mo}^?$ and we also have $(e_1, e_2) \in \text{mo}$, we then have $(e_1, w) \in \text{mo}$, and thus since $e_1 \in W_{\text{wb}} \cup U_{\text{wb}}$ (the assumption of case ii) and $e_1 \notin E^0$, we also have $e_1 \in \mathcal{PW}$. Consequently, from the construction of π we know there exists λ' such that $\lambda' \in \pi$ and $\lambda'=\texttt{genPL}(e_1, G)$; i.e. (from Prop. 3) $\texttt{getPE}(\lambda')=e_1$. Moreover, since $e_1 \in W_{\text{wb}} \cup U_{\text{wb}}$ and $\lambda'=\texttt{genPL}(e_1, G) \in \pi$ from Prop. 3 we know λ_1 and λ' appear immediately next to each other in π: $\pi = -.\lambda_1.\lambda'.-.\lambda_2.-$. On the other hand, since $\texttt{genPL}(e_2, G)=\lambda \in \pi$ and either $e_2 \in W_{\text{wb}} \cup U_{\text{wb}}$ or $e_2 \in ST \setminus (W_{\text{wb}} \cup U_{\text{wb}})$, from Prop. 3 we know $\lambda_2 \preccurlyeq_\pi \lambda$. That is, $\pi = -.\lambda_1.\lambda'.-.\lambda_2.-.\lambda.-$. Consequently we have $\lambda' \prec_\pi \lambda$ and $\texttt{getPE}(\lambda')=e_1$, as required.

RTS (11)

Pick arbitrary $x, y \in \text{Loc}_{\text{wb}}, e_1 \in ST_x, e_2 \in FL_y, \lambda_1, \lambda_2$ such that $(x, y) \in \text{scl}$, $\texttt{getVE}(\lambda_1)=e_1$, $\texttt{getVE}(\lambda_2)=e_2$ and $\lambda_1 \prec_\pi \lambda_2$. From Prop. 3 we then have $\texttt{genVL}(e_1, G)=\lambda_1$ and $\texttt{genVL}(e_2, G)=\lambda_2$. As $x \in \text{Loc}_{\text{wb}}$ and $e_1 \in ST_x$, there are then three cases consider: i) $e_1 \in NTW_{\text{wb}}$; or ii) $e_1 \in W_{\text{wb}} \cup U_{\text{wb}}$ and $\texttt{genPL}(e_1, G) \in \pi$; or iii) $e_1 \in W_{\text{wb}} \cup U_{\text{wb}}$ and $\texttt{genPL}(e_1, G) \notin \pi$.

In case (i), we then simply have $\texttt{genPL}(e_1, G)=\texttt{genVL}(e_1, G)=\lambda_1$ and thus from Prop. 3 we have $\texttt{getPE}(\lambda_1)=e_1$. That is, we have $\texttt{getPE}(\lambda_1)=e_1$ and $\lambda_1 \prec_\pi \lambda_2$, as required.

In case (ii), let $\lambda'=\texttt{genPL}(e_1, G)$ and thus from Prop. 3 we have $\texttt{getPE}(\lambda')=e_1$. Moreover, as $e_1 \in W_{\text{wb}} \cup U_{\text{wb}}$, from Prop. 3 we know λ_1 and λ' appear immediately next to each other in π: $\pi = -.\lambda_1.\lambda'.-.\lambda_2$. That is, we have $\texttt{getPE}(\lambda')=e_1$ and $\lambda' \prec_\pi \lambda_2$, as required.

In case (iii) pick w such that $P(x)=w$. From the assumption of the case and the construction of π we then know that $e_1 \notin \mathcal{PW}$, and thus since mo is total on ST_x, from the definition of $\mathcal{PW}$ we know $(w, e_1) \in \text{mo}$. As G is consistent, we know there exists $w' \in ST_x$ such that $(w', e_2) \in \text{pf}$. Moreover, since G is consistent, from WEAK-PERSIST we know $(w', w) \in \text{mo}^?$ and thus since $(w, e_1) \in \text{mo}$, we also have $(w', e_1) \in \text{mo}$. Consequently as $(w', e_2) \in \text{pf}$, we have $(e_2, e_1) \in \text{pb}$, and thus since G is consistent $(e_2, e_1) \in \text{ob}$. As such, from Prop. 3 we then have $\texttt{genVL}(e_2, G) \prec_\pi \texttt{genVL}(e_1, G)$, and thus from the uniqueness of labels in π (given by Prop. 3) we have $\lambda_2 \prec_\pi \lambda_1$. This, however, leads to a contradiction as we also have $\lambda_1 \prec_\pi \lambda_2$ and $\prec_\pi$ is a strict total order.

RTS (12)

Pick arbitrary $x, y \in \text{Loc}_{\text{wb}}, e_1, e \in ST_x, e_2 \in FO_y, \lambda_1, \lambda_2, \lambda_f$ such that $(x, y) \in \text{scl}$, $\texttt{getVE}(\lambda_1)=e_1$, $\texttt{getVE}(\lambda_2)=e_2$, $\lambda_1 \prec_\pi \lambda_2$, $\lambda_f=\text{P}\langle e_2, e\rangle$ and $\lambda_f \in \pi$. From Prop. 3 we then have $\texttt{genVL}(e_1, G)=\lambda_1$, $\texttt{genVL}(e_2, G)=\lambda_2$, $\texttt{getPE}(\lambda_f)=e_2$, $\lambda_f \in \texttt{genPL}(e_2, G)$ and $\lambda_2 \prec_\pi \lambda_f$. As $x \in \text{Loc}_{\text{wb}}$ and $e_1 \in ST_x$, there are then three cases consider: i) $e_1 \in NTW_{\text{wb}}$; or ii) $e_1 \in W_{\text{wb}} \cup U_{\text{wb}}$ and $\texttt{genPL}(e_1, G) \in \pi$; or iii) $e_1 \in W_{\text{wb}} \cup U_{\text{wb}}$ and $\texttt{genPL}(e_1, G) \notin \pi$.

In case (i), we then simply have $\texttt{genPL}(e_1, G)=\texttt{genVL}(e_1, G)=\lambda_1$ and thus from Prop. 3 we have $\texttt{getPE}(\lambda_1)=e_1$. That is, we have $\texttt{getPE}(\lambda_1)=e_1$ and $\lambda_1 \prec_\pi \lambda_2 \prec_\pi \lambda_f$, and thus $\lambda_1 \prec_\pi \lambda_f$, as required.

In case (ii), let $\lambda'=\texttt{genPL}(e_1, G)$; from Prop. 3 we thus have $\texttt{getPE}(\lambda')=e_1$. Moreover, as $e_1 \in W_{\text{wb}} \cup U_{\text{wb}}$, from Prop. 3 we know λ_1 and λ' appear immediately next to each other in π: $\pi = -.\lambda_1.\lambda'.-.\lambda_2$. That is, we have $\texttt{getPE}(\lambda')=e_1$ and $\lambda' \prec_\pi \lambda_2 \prec_\pi \lambda_f$, and thus $\lambda' \prec_\pi \lambda_f$, as required.

In case (iii) pick w such that $P(x)=w$. From the assumption of the case and the construction of π we then know that $e_1 \notin \mathcal{PW}$, and thus since mo is total on ST_x, from the definition of $\mathcal{PW}$ we know $(w, e_1) \in$ mo. As G is consistent, we know there exists $w' \in ST_x$ such that $(w', e_2) \in$ pf. Moreover, as $\lambda_f \in \texttt{genPL}(e_2, G)$ and $\lambda_f \in \pi$, from the construction of π we know $e_2 \in \mathcal{PFO}$; thus from the definition of $\mathcal{PFO}$ and since $(w', e_2) \in$ pf we know $w' \in \mathcal{PW} \cup NTW$. If $w' \in NTW$, then since G is consistent, from WEAK-PERSIST we know $(w', w) \in \text{mo}^?$; similarly, if $w' \in \mathcal{PW}$, then from the definition of $\mathcal{PW}$ we know $(w', w) \in \text{mo}^?$. In both cases we thus have $(w', w) \in \text{mo}^?$. As $(w, e_1) \in$ mo, we thus have $(w', e_1) \in$ mo; as $(w', e_2) \in$ pf, we also have $(e_2, e_1) \in$ pb. As such, since G is consistent we know $(e_2, e_1) \in$ ob. Therefore, from Prop. 3 we have $\texttt{genVL}(e_2, G) <_\pi \texttt{genVL}(e_1, G)$, and thus from the uniqueness of labels in π (given by Prop. 3) we have $\lambda_2 <_\pi \lambda_1$. This, however, leads to a contradiction as we also have $\lambda_1 <_\pi \lambda_2$ and $<_\pi$ is a strict total order.

RTS (13)
Pick arbitrary $x, y \in \text{Loc}_{\text{wb}}, e_2 \in ST_x, e_1 \in FO_y, \lambda_1, \lambda_2, \lambda$ such that $(x, y) \in$ scl, $\texttt{getVE}(\lambda_1)=e_1$, $\texttt{getVE}(\lambda_2)=e_2$, $\lambda_1 <_\pi \lambda_2$, $\texttt{getPE}(\lambda)=e_2$ and $\lambda \in \pi$. From Prop. 3 we then have $\texttt{genVL}(e_1, G)=\lambda_1$, $\texttt{genVL}(e_2, G)=\lambda_2$, $\texttt{genPL}(e_2, G)=\lambda$ and $\lambda_2 \leqslant_\pi \lambda$. As G is PEx86-consistent, we know there exists $e \in ST_x$ such that $(e, e_1) \in$ pf. Given the construction of π, there are now two cases to consider: either i) $\text{P}\langle e_1, e\rangle \in \pi$, $e \in NTW_x \cup \mathcal{PW}_x$ and $\text{P}\langle e_1, e\rangle$ appears immediately after λ_1 in π; or ii) $\text{P}\langle e_1, e\rangle \notin \pi$ and $e \notin NTW_x \cup \mathcal{PW}_x$.

In case (i), let $\lambda_f=\text{P}\langle e_1, e\rangle$; we then have $\texttt{getPE}(\lambda_f)=e_1$ and thus $\lambda_f \neq \lambda_2$. As such, as $\text{P}\langle e_1, e\rangle$ appears immediately after λ_1 in π, $\lambda_1 <_\pi \lambda_2 \leqslant_\pi \lambda$ and $\lambda_f \neq \lambda_2$, we have $\lambda_f <_\pi \lambda$, $\lambda_f=\text{P}\langle e_1, e\rangle$ and $e \in NTW_x \cup \mathcal{PW}_x \subseteq ST_x$, as required.

In case (ii), as $e \notin NTW_x \cup \mathcal{PW}_x$, from the construction of π we know that $\texttt{genPL}(e, G) \notin \pi$. Moreover, as G is PEx86-consistent and $(e, e_1) \in$ pf, we know $(e, e_1) \in$ ob. As such, since $\texttt{genVL}(e_1, G)=\lambda_1$, from Prop. 3 we know there exists λ_e such that $\texttt{genVL}(e, G)=\lambda_e$ and thus (from Prop. 3) $\texttt{getVE}(\lambda_e)=e$, and $\lambda_e <_\pi \lambda_1$. Consequently, since $\lambda_e <_\pi \lambda_1$ and $\lambda_1 <_\pi \lambda_2$ we have $\lambda_e <_\pi \lambda_2$. On the other hand, as $e, e_2 \in ST_x$, $\texttt{getVE}(\lambda_e)=e$, $\texttt{getVE}(\lambda_2)=e_2$, $\lambda_e <_\pi \lambda_2$ and $\texttt{getPE}(\lambda)=e_2$, from the proof of part (10) we know there exists $\lambda'_e \in \pi$ such that $\texttt{getPE}(\lambda'_e)=e$ and $\lambda'_e <_\pi \lambda$. Consequently, from Prop. 3 we have $\texttt{genVL}(e, G)=\lambda'_e \in \pi$. This, however, contradicts our earlier result that $\texttt{genPL}(e, G) \notin \pi$.

RTS (14)
Pick arbitrary $x, y, z \in \text{Loc}_{\text{wb}}, e_1 \in FO_x, e_2 \in FO_y, e \in ST_z, \lambda_1, \lambda_2, \lambda$ such that $(x, y) \in$ scl, $\texttt{getVE}(\lambda_1)=e_1$, $\texttt{getVE}(\lambda_2)=e_2$, $\lambda_1 <_\pi \lambda_2$, $\lambda=\text{P}\langle e_2, e\rangle$ and $\lambda \in \pi$. We then have $\texttt{getPE}(\lambda)=e_2$, $(y, z) \in$ scl and thus $(x, z) \in$ scl. From Prop. 3 we then have $\texttt{genVL}(e_1, G)=\lambda_1$, $\texttt{genVL}(e_2, G)=\lambda_2$, $\texttt{genPL}(e_2, G)=\lambda$ and $\lambda_2 <_\pi \lambda$. As G is PEx86-consistent, we know there exists $e' \in ST_z$ such that $(e', e_1) \in$ pf. Given the construction of π, there are now two cases to consider: either i) $\text{P}\langle e_1, e'\rangle \in \pi$, $e' \in NTW_z \cup \mathcal{PW}_z$ and $\text{P}\langle e_1, e'\rangle$ appears immediately after λ_1 in π; or ii) $\text{P}\langle e_1, e'\rangle \notin \pi$ and $e' \notin NTW_z \cup \mathcal{PW}_z$.

In case (i), let $\lambda_f=\text{P}\langle e_1, e'\rangle$; we then have $\texttt{getPE}(\lambda_f)=e_1$ and thus $\lambda_f \neq \lambda_2$. As such, as $\text{P}\langle e_1, e'\rangle$ appears immediately after λ_1 in π, $\lambda_1 <_\pi \lambda_2 <_\pi \lambda$ and $\lambda_f \neq \lambda_2$, we have $\lambda_f <_\pi \lambda$, $\lambda_f=\text{P}\langle e_1, e'\rangle$ and $e' \in NTW_z \cup \mathcal{PW}_z \subseteq ST_z$, as required.

In case (ii), as $e' \notin NTW_z \cup \mathcal{PW}_z$, from the construction of π we know that $\texttt{genPL}(e', G) \notin \pi$. Moreover, as G is PEx86-consistent and $(e', e_1) \in$ pf, we know $(e', e_1) \in$ ob. As such, since $\texttt{genVL}(e_1, G)=\lambda_1$, from Prop. 3 we know there exists $\lambda_{e'}$ such that $\texttt{genVL}(e', G)=\lambda_{e'}$ and thus (from Prop. 3) $\texttt{getVE}(\lambda_{e'})=e'$, and $\lambda_{e'} <_\pi \lambda_1$. On the other hand, as $\lambda=\text{P}\langle e_2, e\rangle \in \pi$, we know $(e, e_2) \in$ pf and thus (since G is consistent), $(e, e_2) \in$ ob. Consequently, from Prop. 3 we know there exists λ_e such that $\texttt{genVL}(e, G)=\lambda_e$, $\texttt{getVE}(\lambda_e)=e$, and $\lambda_e <_\pi \lambda_2$. Moreover, since $\lambda=\text{P}\langle e_2, e\rangle \in \pi$, we know that either $e \in NTW$ in which case $\texttt{getPE}(\lambda_e)=\texttt{getVE}(\lambda_e)=e$, or $e \in \mathcal{PW}$ in which case there exists $\lambda^p_e=\texttt{genPL}(e, G)$(and thus from Prop. 3 $\texttt{getPE}(\lambda^p_e)=e$) such that λ^p_e appears immediately after λ_e

in π. That is, in either case we know there exists λ_e^p such that $\mathtt{getPE}(\lambda_e^p)=e$ and $\lambda_e \leqslant_\pi \lambda_e^p$. Since $e, e' \in ST_z$ and G is consistent, we know either: a) $(e', e) \in \mathsf{mo} \subseteq \mathsf{ob}$; or b) $(e, e') \in \mathsf{mo} \subseteq \mathsf{ob}$.

In case (ii.a), since $\mathtt{genVL}(e, G)=\lambda_e$, $\mathtt{genVL}(e', G)=\lambda_{e'}$, from Prop. 3 we know $\lambda_{e'} <_\pi \lambda_e$. As such, since $\mathtt{getPE}(\lambda_e^p)=e$ and $\lambda_e \leqslant_\pi \lambda_e^p$, from the proof of part (10) we know there exists $\lambda_{e'}^p \in \pi$ such that $\mathtt{getPE}(\lambda_{e'}^p)=e'$ and $\lambda_{e'}^p <_\pi \lambda_e^p$. That is, from Prop. 3 we have $\mathtt{genPL}(e', G)=\lambda_{e'}^p \in \pi$, contradicting our earlier result stating $\mathtt{genPL}(e', G) \notin \pi$.

In case (ii.b), since $(e, e') \in \mathsf{mo}$ and $(e, e_2) \in \mathsf{pf}$, we have $(e_2, e') \in \mathsf{pb}$ and thus since G is consistent we have $(e_2, e') \in \mathsf{ob}$. Consequently, since $\mathtt{genVL}(e', G)=\lambda_{e'}$ and $\mathtt{genVL}(e_2, G)=\lambda_2$, from Prop. 3 we have $\lambda_2 <_\pi \lambda_{e'}$. On the other hand, we have $\lambda_{e'} <_\pi \lambda_1$ and $\lambda_1 <_\pi \lambda_2$, and thus $\lambda_{e'} <_\pi \lambda_2$. Since we have both $\lambda_2 <_\pi \lambda_{e'}$ and $\lambda_{e'} <_\pi \lambda_2$, this leads to a contradiction as $<_\pi$ is as strict total order.

RTS (15)
Pick arbitrary $x, y \in \mathrm{Loc}_{\mathsf{wb}}$, $e_1 \in FO_x$, $e_2 \in FL_y$, $e \in ST_z$, λ_1, λ_2, S such that $(x, y) \in \mathrm{scl}$, $\lambda_1 = \mathsf{B}\langle e_1, S\rangle$ and thus $\mathtt{getVE}(\lambda_1)=e_1$, $\mathtt{getVE}(\lambda_2)=e_2$, and $\lambda_1 <_\pi \lambda_2$. From Prop. 3 we then have $\mathtt{genVL}(e_1, G)=\lambda_1$ and $\mathtt{genVL}(e_2, G)=\lambda_2$. Pick an arbitrary $e' \in S$ and let $\mathtt{loc}(e')=z$, i.e. $e' \in ST_z$. Since $\mathtt{genVL}(e_1, G)=\lambda_1$, from the construction of π we know $(e', e_1) \in \mathsf{pf}$ and $(x, z) \in \mathrm{scl}$. As such since $(x, y) \in \mathrm{scl}$ we also have $(y, z) \in \mathrm{scl}$. Given the construction of π, there are now two cases to consider: either i) $\mathsf{P}\langle e_1, e'\rangle \in \pi$, $e' \in NTW_z \cup \mathcal{PW}_z$ and $\mathsf{P}\langle e_1, e'\rangle$ appears immediately after λ_1 in π; or ii) $\mathsf{P}\langle e_1, e'\rangle \notin \pi$ and $e' \notin NTW_z \cup \mathcal{PW}_z$.

In case (i), let $\lambda_f = \mathsf{P}\langle e_1, e'\rangle$; we then have $\mathtt{getPE}(\lambda_f)=e_1$ and thus $\lambda_f \neq \lambda_2$. As such, as $\mathsf{P}\langle e_1, e'\rangle$ appears immediately after λ_1 in π, $\lambda_1 <_\pi \lambda_2$ and $\lambda_f \neq \lambda_2$, we have $\lambda_f <_\pi \lambda_2$, $\lambda_f = \mathsf{P}\langle e_1, e'\rangle$ and $e' \in NTW_z \cup \mathcal{PW}_z \subseteq ST_z$, as required.

In case (ii), as $e' \notin NTW_z \cup \mathcal{PW}_z$, from the construction of π we know that $\mathtt{genPL}(e', G) \notin \pi$. Moreover, as G is PEx86-consistent and $(e', e_1) \in \mathsf{pf}$, we know $(e', e_1) \in \mathsf{ob}$. As such, since $\mathtt{genVL}(e_1, G)=\lambda_1$, from Prop. 3 we know there exists $\lambda_{e'}$ such that $\mathtt{genVL}(e', G)=\lambda_{e'}$ and thus (from Prop. 3) $\mathtt{getVE}(\lambda_{e'})=e'$, and $\lambda_{e'} <_\pi \lambda_1$. On the other hand, as $e_2 \in FL_y$ and $(y, z) \in \mathrm{scl}$, we know there exists $e \in ST_z$ such that we know $(e, e_2) \in \mathsf{pf}$ and thus (as G is consistent), $(e, e_2) \in \mathsf{ob}$. Consequently, from Prop. 3 we know there exists λ_e such that $\mathtt{genVL}(e, G)=\lambda_e$, $\mathtt{getVE}(\lambda_e)=e$, and $\lambda_e <_\pi \lambda_2$. Moreover, since $\lambda_e <_\pi \lambda_2$, $\mathtt{getVE}(\lambda_e)=e$, $\mathtt{getVE}(\lambda_2)=e_2$, from the proofs of parts (11) and (7) we know there exists λ_e^p such that $\mathtt{getPE}(\lambda_e^p)=e$ and $\lambda_e \leqslant_\pi \lambda_e^p <_\pi \lambda_2$. Since $e, e' \in ST_z$ and G is consistent, we know either: a) $(e', e) \in \mathsf{mo} \subseteq \mathsf{ob}$; or b) $(e, e') \in \mathsf{mo} \subseteq \mathsf{ob}$.

In case (ii.a), since $\mathtt{genVL}(e, G)=\lambda_e$, $\mathtt{genVL}(e', G)=\lambda_{e'}$, from Prop. 3 we know $\lambda_{e'} <_\pi \lambda_e$. As such, since $\mathtt{getPE}(\lambda_e^p)=e$ and $\lambda_e \leqslant_\pi \lambda_e^p$, from the proof of part (10) we know there exists $\lambda_{e'}^p \in \pi$ such that $\mathtt{getPE}(\lambda_{e'}^p)=e'$ and $\lambda_{e'}^p <_\pi \lambda_e^p$. That is, from Prop. 3 we have $\mathtt{genPL}(e', G)=\lambda_{e'}^p \in \pi$, contradicting our earlier result stating $\mathtt{genPL}(e', G) \notin \pi$.

In case (ii.b), since $(e, e') \in \mathsf{mo}$ and $(e, e_2) \in \mathsf{pf}$, we have $(e_2, e') \in \mathsf{pb}$ and thus since G is consistent we have $(e_2, e') \in \mathsf{ob}$. Consequently, since $\mathtt{genVL}(e', G)=\lambda_{e'}$ and $\mathtt{genVL}(e_2, G)=\lambda_2$, from Prop. 3 we have $\lambda_2 <_\pi \lambda_{e'}$. On the other hand, we have $\lambda_{e'} <_\pi \lambda_1$ and $\lambda_1 <_\pi \lambda_2$, and thus $\lambda_{e'} <_\pi \lambda_2$. Since we have both $\lambda_2 <_\pi \lambda_{e'}$ and $\lambda_{e'} <_\pi \lambda_2$, this leads to a contradiction as $<_\pi$ is as strict total order.

RTS (16)
Pick arbitrary $e_1 \in FO$, $e_2 \in MF \cup SF \cup U$, λ_1, λ_2, S such that $\mathtt{tid}(e_1)=\mathtt{tid}(e_2)$, $\lambda_1 = \mathsf{B}\langle e_1, S\rangle$, $\lambda_1 <_\pi \lambda_2$, $\mathtt{getVE}(\lambda_2)=e_2$. That is, from Prop. 3 we have $\mathtt{genVL}(e_1, G)=\lambda_1$, $\mathtt{getVE}(\lambda_1)=e_1$, $\mathtt{genVL}(e_2, G)=\lambda_2$. Moreover, from the definition of $\mathtt{genVL}(.,.)$ we know $S = \left\{w \,\middle|\, (w, e_1) \in \mathsf{pf}\right\}$. As $\mathtt{tid}(e_1)=\mathtt{tid}(e_2)$, we know either $(e_1, e_2) \in \mathrm{po}$ or $(e_2, e_1) \in \mathrm{po}$. Moreover, as $e_1 \in FO$, $e_2 \in MF \cup SF \cup U$, $([FO]; \mathrm{po}; [MF \cup SF \cup U]) \cup ([MF \cup SF \cup U]; \mathrm{po}; [FO]) \subseteq \mathsf{ppo}(\mathrm{po}) \subseteq \mathsf{ob}$, and $\mathtt{genVL}(e_1, G) <_\pi \mathtt{genVL}(e_2, G)$, from Prop. 3 we have $(e_1, e_2) \in \mathrm{po}$.

Pick an arbitrary $w \in S$. As λ_1=genVL(e_1, G)=B$\langle e_1, S\rangle$ and $w \in S$, we know $(w, e_1) \in$ pf and thus since G is consistent we know $(w, e_1) \in$ ob. As such, since genVL(e_1, G)=λ_1, from Prop. 3 we know there exists λ_w=genVL(w, G) such that $\lambda_w <_\pi \lambda_1$. We next demonstrate that for this arbitrary w we have $w \in NTW \cup \mathcal{PW}$.

Let $\texttt{loc}(w)$=x and pick w_m such that $P(x)$=w_m. As $(w, e_1) \in$ pf, we know that $x \in \text{Loc}_{\text{wb}}$ and thus either $w \in NTW_{\text{wb}}$, or $w \in W_{\text{wb}} \cup U_{\text{wb}}$. In the former case we then have $w \in NTW$ and thus $w \in NTW \cup \mathcal{PW}$. In the latter case, since $(e_1, e_2) \in$ po, $(w, e_1) \in$ pf, $e_1 \in FO$, $e_2 \in MF \cup SF \cup U$ and G is consistent, from WEAK-PERSIST we know $(w, w_m) \in \text{mo}^?$. As such, from the definition of $\mathcal{PW}$ we have $w \in \mathcal{PW}$ and thus $w \in NTW \cup \mathcal{PW}$.

We thus demonstrated that for an arbitrary $w \in S$, we have $w \in NTW \cup \mathcal{PW}$. Consequently, as S=$\{w \mid (w, e_1) \in \text{pf}\}$, from the definition of $\mathcal{PFO}$ we know $e_1 \in \mathcal{PFO}$. As such, from the construction of π we know there exist an enumeration $[w_1 \cdots w_n]$ of S and π' such that $\pi' \triangleq \text{P}\langle e_1, w_1\rangle. \cdots .\text{P}\langle e_1, w_n\rangle$, and λ_1=genVL(e_1, G) and π' are adjacent in π: $\pi \triangleq -.\lambda_1.\pi'.-.\lambda_2.-$. That is, since $w \in S$, we know $\pi \triangleq -.\lambda_1.-\text{P}\langle e_1, w\rangle.-.\lambda_2.-$, and thus $\text{P}\langle e_1, w\rangle <_\pi \lambda_2$. □

Proposition 4. *Let G denote an PEx86 consistent execution of program* P. *Let $e_1 \cdots e_n$ denote an enumeration of $G.(E \setminus E^0)$ that respects G.po. Then there exist $\text{P}_1 \cdots \text{P}_n$ and $\text{P}_0 \triangleq \text{P}$ such that for all $i \in \{1 \cdots n\}$:*

$$\text{P}_{i-1} \left(\xrightarrow{\mathcal{E}\langle - \rangle}\right)^* \xrightarrow{\text{genL}(e_i, G)} \left(\xrightarrow{\mathcal{E}\langle - \rangle}\right)^* \text{P}_i$$

Definition 15 (Graph operational semantics).

$$\frac{\text{P} \xrightarrow{\mathcal{E}\langle \tau \rangle} \text{P}' \qquad \text{wfp}(\pi)}{\text{P}, \pi \Rightarrow \text{P}', \pi}\ \text{G-SilentP}$$

$$\frac{\lambda \in \{\text{B}\langle e\rangle, \text{B}\langle e, -\rangle, \text{P}\langle e\rangle, \text{P}\langle e, -\rangle\} \qquad \text{fresh}(\lambda, \pi) \qquad \text{wfp}(\pi) \qquad \text{wfp}(\pi.\lambda)}{\text{P}, \pi \Rightarrow \text{P}, \pi.\lambda}\ \text{G-Prop}$$

$$\frac{\text{P} \xrightarrow{\lambda} \text{P}' \qquad \lambda \neq \mathcal{E}\langle - \rangle \qquad \text{fresh}(\lambda, \pi) \qquad \text{wfp}(\pi) \qquad \text{wfp}(\pi.\lambda)}{\text{P}, \pi \Rightarrow \text{P}', \pi.\lambda}\ \text{G-Step}$$

Lemma 7. *Given a program* P, *for all PEx86-consistent executions G of* P *and all π, if* $\texttt{getPath}(G)$=π, *then there exists* P′ *such that* $\text{P}, \epsilon \Rightarrow^* \text{P}', \pi$.

Proof. Pick arbitrary program P, PEx86-consistent execution G of P and π such that $\texttt{getPath}(G) = \pi$. From Prop. 3 we know π respects G.po. That is, π is of the form: $\text{genL}(e_1, G).s_1. \cdots .\text{genL}(e_m, G).s_m$, where:

(i) $e_1 \cdots e_m$ is an enumeration of $G.E$ respecting G.po (if $(e, e') \in G$.po then $\text{genL}(e, G) <_\pi \text{genL}(e', G)$).

(ii) For each $j \in \{i \cdots m\}$, $s_j = \lambda_{(j,1)}. \cdots .\lambda_{(j,k_j)}$ and each $\lambda_{(j,r)}$ is of the form $\text{B}\langle -\rangle$ or $\text{B}\langle -, -\rangle$ or $\text{P}\langle -\rangle$ or $\text{P}\langle -, -\rangle$.

Moreover, from Lemma 6 we know wfp(π) holds and thus:

$$\forall \lambda, p, q.\ \pi = p.\lambda.q \Rightarrow \text{fresh}(\lambda, p.q) \tag{17}$$

There are now two cases to consider: 1) $m = 0$; or 2) $m > 0$. In case (1), we then have $\pi = \epsilon$ and we trivially have $\text{P}, \epsilon \Rightarrow^* \text{P}, \epsilon$, as required.

In case (2) from Prop. 4 we know there exists $\text{P}_1 \cdots \text{P}_m$ and P_0=P such that for $j \in \{1 \cdots m\}$:

$$\text{P}_{j-1} \left(\xrightarrow{\mathcal{E}\langle - \rangle}\right)^* \xrightarrow{\text{genL}(e_j, G)} \left(\xrightarrow{\mathcal{E}\langle - \rangle}\right)^* \text{P}_j \tag{18}$$

For $j \in \{1 \cdots m\}$, from (18) we know there exist $\mathsf{P}'_j, \mathsf{P}''_j$ such that $\mathsf{P}_{j-1}(\xrightarrow{\mathcal{E}\langle - \rangle})^* \mathsf{P}'_j \xrightarrow{\mathsf{genL}(e_j, G)} \mathsf{P}''_j(\xrightarrow{\mathcal{E}\langle - \rangle})^* \mathsf{P}_j$. Let $\pi_j = \mathsf{genL}(e_1, G).s_1. \cdots .s_j.\mathsf{genL}(e_j, G).s_j$, for $j \in \{1 \cdots m\}$. As $\mathsf{wfp}(\pi)$ holds, from Prop. 1 we have:

$$\forall j \in \{1 \cdots m\}.\ \mathsf{wfp}(\pi_j) \tag{19}$$

As such, from G-SilentP, G-Step, G-Prop, (17) and (19) we then have:

$$\begin{array}{ll} & \mathsf{P}_{j-1}, \pi_{j-1} \\ \Rightarrow^* & \mathsf{P}'_j, \pi_{j-1} \\ \Rightarrow & \mathsf{P}''_j, \pi_{j-1}.\mathsf{genL}(e_j, G) \\ \Rightarrow^* & \mathsf{P}_j, \pi_{j-1}.\mathsf{genL}(e_j, G) \\ \Rightarrow & \mathsf{P}_j, \pi_j \end{array}$$

Consequently, we have:

$$\mathsf{P}_0, \epsilon \Rightarrow^* \mathsf{P}_1, \pi_1 \Rightarrow^* \cdots \Rightarrow^* \mathsf{P}_m, \pi_m$$

That is, as $\mathsf{P}_0 = \mathsf{P}$ and $\pi_m = \pi$, we have $\mathsf{P}, \epsilon \Rightarrow^* \mathsf{P}_m, \pi$, as required. □

Lemma 8. *For all $\pi, \lambda, M, PB, B, e, \tau$, if $\mathsf{wfp}(\pi.\lambda)$, $\mathsf{wf}(M, PB, B, \pi)$ and $\mathtt{tid}(e)=\tau$, then:*

(1) $\mathtt{getVE}(\lambda)=e \Rightarrow \forall e' \in B(\tau).\ (e', e) \notin \mathsf{PPO}(B(\tau))$
(2) $\mathtt{getVE}(\lambda)=e \wedge e \in MF \cup U \cup R_{\mathsf{nc}} \Rightarrow B(\tau)=\epsilon$
(3) $\mathtt{getVE}(\lambda)=e \wedge e \in NTW_{\mathsf{wb}} \Rightarrow PB(\mathtt{loc}(e))=\epsilon$
(4) $\forall w.\ (\lambda=\mathsf{R}\langle e, w\rangle \vee \lambda=\mathsf{U}\langle e, w\rangle) \Rightarrow w=\mathsf{rd}(M, pb, B(\tau), \mathtt{loc}(r))$

$$\text{where } pb \triangleq \begin{cases} PB(\mathtt{loc}(r)) & \text{if } \mathtt{loc}(r) \in Loc_{\mathsf{wb}} \\ \epsilon & \text{otherwise} \end{cases}$$

(5) $\forall w.\ (\mathtt{loc}(e) \in Loc_{\mathsf{nc}} \wedge \lambda=\mathsf{R}\langle e, w\rangle) \vee (\mathtt{loc}(e) \notin Loc_{\mathsf{wb}} \wedge \lambda=\mathsf{U}\langle e, w\rangle) \Rightarrow w = M(\mathtt{loc}(e))=w$
(6) $\forall S.\ \lambda=\mathsf{P}\langle e, S\rangle \wedge e \in FL \Rightarrow S = \big\{M(x) \bigm| (x, \mathtt{loc}(e)) \in \mathsf{scl}\big\} \wedge (\forall x.\ (x, \mathtt{loc}(e)) \in \mathsf{scl} \Rightarrow PB(x)=\epsilon)$
(7) $\forall S.\ \lambda=\mathsf{B}\langle e, S\rangle \wedge e \in FO \Rightarrow S = \big\{\mathsf{rd}(M, PB(x), \epsilon, x) \bigm| (x, \mathtt{loc}(e)) \in \mathsf{scl}\big\}$
(8) $\mathtt{getVE}(\lambda)=e \wedge e \in MF \cup SF \cup U \Rightarrow \forall x.\ PB(x) \cap FO_\tau=\emptyset$
(9) $\mathtt{getPE}(\lambda)=e \wedge e \in W_{\mathsf{wb}} \cup U_{\mathsf{wb}} \Rightarrow PB(\mathtt{loc}(e)) = e.-$
(10) $\forall w.\ \lambda=\mathsf{P}\langle e, w\rangle \wedge e \in FO \Rightarrow PB(\mathtt{loc}(e)) = e.- \wedge M(\mathtt{loc}(e))=w$

Proof. Pick arbitrary $\pi, \lambda, M, PB, B, e, \tau$ such that $\mathsf{wfp}(\pi)$, $\mathsf{wfp}(\pi.\lambda)$, $\mathsf{wf}(M, PB, B, \pi)$, $\mathtt{getVE}(\lambda)=e$ and $\mathtt{tid}(e)=\tau$. We prove each part in turn.

RTS (1)
Let $\mathtt{getVE}(\lambda)=e$. Pick an arbitrary $e' \in B(\tau)$. Let us proceed by contradiction and assume that $(e', e) \in \mathsf{PPO}(B(\tau))$. As $(e', e) \in \mathsf{PPO}(B(\tau))$ by definition we know $(e', e) \in \mathsf{PO}(B(\tau))$ and thus from the definition of $\mathsf{PO}(.)$ we know: $e, e' \in B(\tau)$. As such, since $\mathsf{wf}(M, PB, B, \pi)$ and thus $B(\tau)=\mathtt{buff}(\pi, \tau)$, from the definition of $\mathtt{buff}(.,.)$ we know that $\forall \lambda' \in \pi.\ \mathtt{getVE}(\lambda') \neq e \wedge \mathtt{getVE}(\lambda') \neq e'$. On the other hand, from Prop. 2 we know $\mathsf{PO}(B(\tau)) \subseteq \mathsf{PO}(\pi)$, and thus $(e', e) \in \mathsf{PO}(\pi)$. That is, there exist $\lambda_{e'}, \lambda_e$ such that $\mathtt{getE}(\lambda_{e'})=e'$, $\mathtt{getE}(\lambda_e)=e$ and $\lambda_{e'} <_\pi \lambda_e$. Moreover, as $(e', e) \in \mathsf{PO}(\pi)$, from the uniqueness of labels in π (given by $\mathsf{wfp}(\pi)$) we know $e \neq e'$. Consequently, as $\forall \lambda' \in \pi.\ \mathtt{getVE}(\lambda') \neq e \wedge \mathtt{getVE}(\lambda') \neq e'$, $e \neq e'$ and $\mathtt{getVE}(\lambda)=e$, we also know $\forall \lambda' \in \pi.\lambda.\ \mathtt{getVE}(\lambda') \neq e'$.

Additionally, from Prop. 1 we know $\mathsf{PPO}(B(\tau)) \subseteq \mathsf{PPO}(\pi)$ and that $\mathsf{PPO}(\pi) \subseteq \mathsf{PPO}(\pi.\lambda)$; i.e. $\mathsf{PPO}(B(\tau)) \subseteq \mathsf{PPO}(\pi.\lambda)$ and thus $(e', e) \in \mathsf{PPO}(\pi.\lambda)$. Consequently, since $(e', e) \in \mathsf{PPO}(\pi.\lambda)$, $\mathtt{getVE}(\lambda)=e$, $\lambda \in \pi.\lambda$ and $\mathsf{wfp}(\pi.\lambda)$, from the definition of $\mathsf{wfp}()$ we know there exists λ' such that $\lambda' <_{\pi.\lambda} \lambda$ and $\texttt{getVE}(\lambda')=e'$. That is, there exists $\lambda' \in \pi.\lambda$ such that $\texttt{getVE}(\lambda')=e'$. This however contradicts our earlier result that $\forall \lambda' \in \pi.\lambda.\ \texttt{getVE}(\lambda') \neq e'$.

RTS (2)
Assume $\texttt{getVE}(\lambda)=e$ and $e \in MF \cup U \cup R_{\text{nc}}$. Let us proceed by contradiction and assume that there exists $e' \in \text{BEvent}$ such that $e' \in B(\tau)$. We then know that $\texttt{tid}(e') = \tau$. From the definition of $\texttt{wf}(M, PB, B, \pi)$ we then know there exist $\lambda' \in \pi$ such that $\texttt{getE}(\lambda')=e'$, and for all $\lambda'' \in \pi$, $\texttt{getVE}(\lambda'') \neq e'$. As $\lambda' \in \pi$, we have $\lambda' <_{\pi.\lambda} \lambda$. Moreover, since $\lambda' <_{\pi.\lambda} \lambda$, $e \in MF \cup U \cup R_{\text{nc}}$ and $\texttt{tid}(e') = \tau$, we have $(e', e) \in \texttt{PO}(\pi.\lambda)$ and by definition of ppo we also have $(e', e) \in \texttt{PPO}(\pi.\lambda)$. Consequently, since $\texttt{wfp}(\pi.\lambda)$ holds, from the definition of $\texttt{wfp}(.)$ we know there exists λ'' such that $\texttt{getVE}(\lambda'')=e'$ and $\lambda' <_{\pi.\lambda} \lambda$. That is, there exists $\lambda'' \in \pi$ such that $\texttt{getVE}(\lambda'')=e'$. This however leads to a contradiction as earlier we established that for all $\lambda'' \in \pi$, $\texttt{getVE}(\lambda'') \neq e'$. We can thus conclude that $B(\tau) = \epsilon$.

RTS (3)
Assume $\texttt{getVE}(\lambda)=e$ and $e \in NTW_{\text{wb}}$. As such, we also have $\texttt{getPE}(\lambda)=e$. Let $\texttt{loc}(e)=x \in \text{Loc}_{\text{wb}}$. Let us proceed by contradiction and assume that there exists some $e' \in PB(x)$. From the definition of $\texttt{wf}(M, PB, B, \pi)$ we then know there exist $\lambda' \in \pi$ such that either i) $e' \in \text{PBEvent} \cap ST_x$, $\texttt{getVE}(\lambda')=e'$ and $\texttt{P}\langle e' \rangle \notin \pi$, i.e. $\forall \lambda'' \in \pi.\ \texttt{getPE}(\lambda'') \neq e'$; or ii) there exists S such that $\lambda'=\texttt{B}\langle e', S\rangle$ (i.e. $\texttt{getVE}(\lambda')=e'$ and $e' \in FO$), $\forall w.\ \texttt{loc}(w)=x \Rightarrow \texttt{P}\langle e', w\rangle \notin \pi$, and that (from the types of ALabels) there exists $y \in \text{Loc}_{\text{wb}}$ such that $(x, y) \in \text{scl}$ and $\texttt{loc}(e')=y$. As $\lambda' \in \pi$, we have $\lambda' <_{\pi.\lambda} \lambda$.

In case (i), since $e' \in \text{PBEvent} \cap ST_x$, $e \in NTW_x$, $\lambda' <_{\pi.\lambda} \lambda$, $\texttt{getVE}(\lambda')=e'$, $\texttt{getVE}(\lambda)=e$, and $\texttt{getPE}(\lambda)=e \in \pi$, from $\texttt{wfp}(\pi.\lambda)$ we know there exists λ'' such that $\texttt{getPE}(\lambda'') = e'$ and $\lambda'' <_{\pi.\lambda} \lambda$. That is, there $\lambda'' \in \pi$ such that $\texttt{getPE}(\lambda'') = e'$. This however contradicts the assumption of case (i) stating $\forall \lambda'' \in \pi.\ \texttt{getPE}(\lambda'') \neq e'$.

In case (ii), since $x, y \in \text{Loc}_{\text{wb}}$, $(x, y) \in \text{scl}$, $e' \in FO_y$, $e \in NTW_x$, $\texttt{getVE}(\lambda')=e'$, $\texttt{getVE}(\lambda)=e$, $\lambda' <_{\pi.\lambda} \lambda$, $\texttt{getPE}(\lambda)=e$ and $\lambda \in \pi.\lambda$, from $\texttt{wfp}(\pi.\lambda)$ we know there exists $w \in ST_x$ such that $\texttt{P}\langle e', w\rangle <_{\pi.\lambda} \lambda$. That is, there exists w such that $\texttt{loc}(w)=x$ and $\texttt{P}\langle e', w\rangle \in \pi$. This however contradicts the assumption of case (ii) stating $\forall w.\ \texttt{loc}(w)=x \Rightarrow \texttt{P}\langle e', w\rangle \notin \pi$.
We can thus conclude that $PB(x) = \emptyset$.

RTS (4)
Pick an arbitrary w such that $\lambda=\texttt{R}\langle e, w\rangle \vee \lambda=\texttt{U}\langle e, w\rangle$. Let $\texttt{loc}(e)=x$, $B(\tau)=b$ and pb be as defined in the premise. From the definition of $\texttt{wfp}(\pi.\lambda)$ we know that $\texttt{wfrd}(e, w, \pi)$ holds, i.e. $\texttt{lread}(\pi, x, \tau)=w$. As such, from the definition of $\texttt{lread}(\pi, x, \tau)$ there are now three cases:

i) $\exists \pi_1, \pi_2, \lambda_w.\ w \in ST_x \wedge \pi=\pi_1.\lambda_w.\pi_2 \wedge \texttt{getE}(\lambda_w)=w \wedge \texttt{tid}(w)=\tau$
$\wedge \forall \lambda' \in \pi.\ \texttt{getVE}(\lambda') \neq w$
$\wedge \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \texttt{getE}(\lambda')=e' \wedge \texttt{tid}(e')=\tau\right\} = \emptyset$

ii) the previous condition does not holds and:
$\exists \pi_1, \pi_2, \lambda_w.\ w \in ST_x \wedge \pi=\pi_1.\lambda_w.\pi_2 \wedge \texttt{getVE}(\lambda_w)=w$
$\wedge \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \texttt{getVE}(\lambda')=e'\right\} = \emptyset$

iii) the previous two conditions do not hold and $w=init_x$

In case (i), since $\texttt{wf}(M, pb, b, \pi)$ holds, from its definition we know there exists b_1, b_2 such that $b = b_1.w.b_2$ and $\forall e' \in b_2 \cap ST.\ \texttt{loc}(e') \neq x$. As such, from the definition of $\texttt{rd}(.,.,.)$ we know the value will be read from b and that $\texttt{rd}(M, pb, b, x) = w$, as required.

In case (ii), there are two cases to consider: a) $x \in \text{Loc}_{\text{wb}}$; or b) $x \notin \text{Loc}_{\text{wb}}$. In case (ii.a), since $\texttt{wf}(M, pb, b, \pi)$ holds, from its definition we know that for all $e' \in b \cap ST$, $\texttt{loc}(e') \neq x$; and that there exists pb_1, pb_2 such that $pb = pb_1.w.pb_2$, and for all $e' \in pb_2 \cap ST$, $\texttt{loc}(e') \neq x$. As such, by definition we have $\texttt{rd}(M, pb, b, x) = w$, as required.

In case (ii.b), since $\texttt{wf}(M, pb, b, \pi)$ holds, from its definition we know that for all $e' \in b \cap ST$, $\texttt{loc}(e') \neq x$. Moreover, from the assumption of the case (ii) and the assumption of case (b) (i.e. since $x \notin \text{Loc}_{\text{wb}}$) we also know: $\texttt{getPE}(\lambda_w){=}w \wedge \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \texttt{getPE}(\lambda'){=}e'\right\} = \emptyset$. That is, $\texttt{pread}(\pi, x){=}w$. Moreover, since $\texttt{wf}(M, pb, b, \pi)$ holds, we know $M(x){=}\texttt{pread}(\pi, x)$, and thus $M(x){=}w$. As such, since $pb = \epsilon$ and $e' \in b \cap ST$, $\texttt{loc}(e') \neq x$, from the definition of $\texttt{rd}(., ., .)$ we have $\texttt{rd}(M, pb, b, x) = M(x) = w$, as required.

In case (iii), since $\texttt{wf}(M, pb, b, \pi)$ holds, from its definition we know for all $e' \in (b \cup pb) \cap ST$, $\texttt{loc}(e') \neq x$; and that $M(x) = init_x$. As such, by definition we have $\texttt{rd}(M, pb, b, x) = w$.

RTS (5)
Pick arbitrary w such that $(\texttt{loc}(e) \in \text{Loc}_{\text{nc}} \wedge \lambda{=}\text{R}\langle e, w\rangle) \vee (\texttt{loc}(e) \notin \text{Loc}_{\text{wb}} \wedge \lambda{=}\text{U}\langle e, w\rangle)$. Let $\texttt{loc}(e){=}x$. From the definition of $\texttt{getVE}(.)$ we then have $(e \in R_{\text{nc}} \wedge \texttt{getVE}(\lambda){=}e) \vee (e \in U \wedge \texttt{getVE}(\lambda){=}e)$. As such, from the proof of part (2) we know $B(\tau){=}\epsilon$. Moreover, since either $x \in \text{Loc}_{\text{nc}}$ or $x \notin \text{Loc}_{\text{wb}}$, we know $x \notin \text{Loc}_{\text{wb}}$. As such, since $B(\tau){=}\epsilon$, from the proof of part (4) we have $w{=}\texttt{rd}(M, \epsilon, \epsilon, x)$. Consequently, from the definition of $\texttt{rd}(., ., .)$ we have $w{=}M(x)$, as required.

RTS (6)
Pick an arbitrary S such that $\lambda{=}\text{P}\langle e, S\rangle$ and let $\texttt{loc}(e){=}y \in \text{Loc}_{\text{wb}}$. We then have $\texttt{getVE}(\lambda){=}\texttt{getPE}(\lambda){=}e$. We first demonstrate that $\forall x.\ (x, y) \in \text{scl} \Rightarrow PB(x){=}\epsilon$. Let us proceed by contradiction and assume there exists $x \in \text{Loc}_{\text{wb}}$ and e' such that $(x, y) \in \text{scl}$ and $e' \in PB(x)$. From the definition of $\texttt{wf}(M, PB, B, \pi)$ we then know there exist $\lambda' \in \pi$ such that either i) $e' \in \text{PBEvent} \cap ST_x$, $\texttt{getVE}(\lambda'){=}e'$ and $\text{P}\langle e'\rangle \notin \pi$, i.e. $\forall \lambda'' \in \pi.\ \texttt{getPE}(\lambda'') \neq e'$; or ii) there exists S' such that $\lambda'{=}\text{B}\langle e', S'\rangle$ (i.e. $\texttt{getVE}(\lambda'){=}e'$ and $e' \in FO$), $\forall w.\ \texttt{loc}(w){=}x \Rightarrow \text{P}\langle e', w\rangle \notin \pi$, and that (from the types of ALabels) there exists $z \in \text{Loc}_{\text{wb}}$ such that $(x, z) \in \text{scl}$ and $\texttt{loc}(e'){=}z$. As $\lambda' \in \pi$, we have $\lambda' <_{\pi.\lambda} \lambda$.

In case (i), since $e' \in \text{PBEvent} \cap ST_x$, $e \in NTW_x$, $\lambda' <_{\pi.\lambda} \lambda$, $\texttt{getVE}(\lambda'){=}e'$, $\texttt{getVE}(\lambda){=}e$, and $\texttt{getPE}(\lambda){=}e \in \pi$, from $\texttt{wfp}(\pi.\lambda)$ we know there exists λ'' such that $\texttt{getPE}(\lambda'') = e'$ and $\lambda'' <_{\pi.\lambda} \lambda$. That is, there $\lambda'' \in \pi$ such that $\texttt{getPE}(\lambda'') = e'$. This however contradicts the assumption of case (i) stating $\forall \lambda'' \in \pi.\ \texttt{getPE}(\lambda'') \neq e'$.

In case (ii), since $x, z \in \text{Loc}_{\text{wb}}$, $(x, y) \in \text{scl}$, $e' \in FO_y$, $e \in NTW_x$, $\texttt{getVE}(\lambda'){=}e'$, $\texttt{getVE}(\lambda){=}e$, $\lambda' <_{\pi.\lambda} \lambda$, $\texttt{getPE}(\lambda){=}e$ and $\lambda \in \pi.\lambda$, from $\texttt{wfp}(\pi.\lambda)$ we know there exists $w \in ST_x$ such that $\text{P}\langle e', w\rangle <_{\pi.\lambda} \lambda$. That is, there exists w such that $\texttt{loc}(w){=}x$ and $\text{P}\langle e', w\rangle \in \pi$. This however contradicts the assumption of case (ii) stating $\forall w.\ \texttt{loc}(w){=}x \Rightarrow \text{P}\langle e', w\rangle \notin \pi$.
We can thus conclude that $PB(x) = \emptyset$.

We next demonstrate that $S{=}\left\{M(x) \,\middle|\, (x, y) \in \text{scl}\right\}$. For each location x such that $(xe, y) \in \text{scl}$, let us write $S(x)$ for the unique write in S on x – note that such a unique write always exists given the type constraints on ALabels. Pick an arbitrary x and let $S(x) = w$; it then suffices to show that $w{=}\texttt{rd}(M, PB(x), \epsilon, x)$. That is, as we previously established that $PB(x){=}\epsilon$, from the definition of $\texttt{rd}(., ., .)$ it suffices to show that $w{=}M(x)$. Moreover, as $\texttt{wf}(M, pb, b, \pi)$ holds, from its definition we know $M(x){=}\texttt{pread}(\pi, x)$ and thus we must show $w{=}\texttt{pread}(\pi, x)$. Finally, from the definition of $\texttt{wfp}(\pi.\lambda)$ we know that $\texttt{wfrd}(e, w, \pi)$ holds, i.e. $\texttt{pread}(\pi, x){=}w$, as required.

The proof of part (7) is analogous to that of part (6) and thus omitted.

RTS (8)
Assume $\mathtt{getVE}(\lambda)=e$ and $e \in MF \cup SF \cup U$. Let us proceed by contradiction and assume that there exists x and $e' \in FO_\tau$ (i.e. $\mathtt{tid}(e')=\tau$) such that $e' \in PB(x)$. From the definition of $\mathsf{wf}(M, PB, B, \pi)$ we then know there exist $\lambda' \in \pi, S$ such that $\lambda'=\mathsf{B}\langle e', S\rangle$ and for all w, $\mathtt{loc}(w)=x \Rightarrow \mathsf{P}\langle e', w\rangle \notin \pi$. As $\lambda' \in \pi$, we have $\lambda' <_{\pi.\lambda} \lambda$. On the other hand, since $\lambda' <_{\pi.\lambda} \lambda$, $e \in MF \cup SF \cup U$, $\mathtt{tid}(e') = \tau$ and $\mathsf{wfp}(\pi.\lambda)$ holds, from the definition of $\mathsf{wfp}(.)$ and the types of ALabels we know there exists $w \in S$ such that $\mathtt{loc}(w)=x$ and $\mathsf{P}\langle e', w\rangle <_{\pi.\lambda} \lambda$. This however leads to a contradiction as earlier we established for all w, $\mathtt{loc}(w)=x \Rightarrow \mathsf{P}\langle e', w\rangle \notin \pi$. We can thus conclude that $\forall x.\ PB(x) \cap FO_\tau = \emptyset$.

RTS (9)
Assume $\mathtt{getPE}(\lambda)=e$ and $e \in W_{\mathsf{wb}} \cup U_{\mathsf{wb}}$. Let $\mathtt{loc}(e)=x$. As $\mathsf{wf}(M, PB, B, \pi)$, $\mathsf{wfp}(\pi)$ and $\mathsf{wfp}(\pi.\lambda)$ hold and $\mathtt{getPE}(\lambda)=e$, we then know $e \in PB(x)$. We next show that e is at the head of $PB(x)$. Let us proceed by contradiction and assume that there exists $e' \in FO \cup W_{\mathsf{wb}} \cup U_{\mathsf{wb}}$ such that $PB(x)=e'.e.-$. From the definition of PB we then know that either $e' \in \text{PBEvent}_x \cap ST$ or there exists y such that $e' \in FO_y$ and $(x, y) \in \mathsf{scl}$. Moreover, since $PB(x)=e'.e.-$, from the definition of $\mathsf{wf}(M, PB, B, \pi)$ we then know there exist $\lambda_e, \lambda_{e'} \in \pi$ such that $\lambda_{e'} <_\pi \lambda_e$, $\mathtt{getVE}(\lambda_e)=e$, $\mathtt{getVE}(\lambda_{e'})=e'$ and either i) $e' \in \text{PBEvent}_x \cap ST$ and $\mathsf{P}\langle e'\rangle \notin \pi$, i.e. $\forall \lambda'' \in \pi.\ \mathtt{getPE}(\lambda'') \neq e'$; or ii) $e' \in FO_y$, $(x, y) \in \mathsf{scl}$, $\lambda'=\mathsf{B}\langle e', -\rangle$ and $\forall w.\ \mathtt{loc}(w)=x \Rightarrow \mathsf{P}\langle e', w\rangle \notin \pi$. As $\lambda_{e'} \in \pi$, we have $\lambda_{e'} <_{\pi.\lambda} \lambda$.

In case (i), since $\lambda_{e'} <_\pi \lambda_e$ and thus $\lambda_{e'} <_{\pi.\lambda} \lambda_e$, $\mathtt{getVE}(\lambda_e)=e$, $\mathtt{getVE}(\lambda_{e'})=e'$, $\mathtt{getPE}(\lambda)=e$, $\lambda \in \pi.\lambda$ and $e, e' \in \text{PBEvent}_x \cap ST \subseteq ST_x$, from $\mathsf{wfp}(\pi.\lambda)$ we know there exists $\lambda' \in \pi.\lambda$ such that $\mathtt{getPE}(\lambda')=e'$ and $\lambda' <_{\pi.\lambda} \lambda$. That is, there exists $\lambda' \in \pi$ such that $\mathtt{getPE}(\lambda')=e'$. This, however, contradicts the assumption of case (i) stating $\forall \lambda'' \in \pi.\ \mathtt{getPE}(\lambda'') \neq e'$.

In case (ii), since $\lambda_{e'} <_\pi \lambda_e$ and thus $\lambda_{e'} <_{\pi.\lambda} \lambda_e$, $\mathtt{getVE}(\lambda_e)=e$, $\mathtt{getVE}(\lambda_{e'})=e'$, $\mathtt{getPE}(\lambda)=e$, $\lambda \in \pi.\lambda$ and $e \in \text{PBEvent}_x \cap ST \subseteq ST_x$, $e' \in FO_y$ and $(x, y) \in \mathsf{scl}$ from $\mathsf{wfp}(\pi.\lambda)$ we know there exists $\lambda' \in \pi.\lambda, w$ such that $\mathtt{loc}(w)=x$, $\lambda'=\mathsf{P}\langle e', w\rangle$ and $\lambda' <_{\pi.\lambda} \lambda$. That is, there exists w such that $\mathtt{loc}(w)=x$ and $\mathsf{P}\langle e', w\rangle \in \pi$. This, however, contradicts the assumption of case (ii) stating $\forall w.\ \mathtt{loc}(w)=x \Rightarrow \mathsf{P}\langle e', w\rangle \notin \pi$.

The proof of part (10) is analogous to that of part (9) and thus omitted. □

Lemma 9. *For all* $\mathsf{P}, \mathsf{P}', \pi, \pi', M, PB, B$, *if* $\mathsf{P}, \pi \Rightarrow \mathsf{P}', \pi'$ *and* $\mathsf{wf}(M, PB, B, \pi)$, *then there exist* M', PB', B' *such that:*

$$\mathsf{P}, M, PB, B, \pi \Rightarrow^* \mathsf{P}', M', PB', B', \pi'$$

Proof. Pick arbitrary $\mathsf{P}, \mathsf{P}', \pi, \pi', M, PB, B$ such that $\mathsf{P}, \pi \Rightarrow \mathsf{P}', \pi'$ and $\mathsf{wf}(M, PB, B, \pi)$. We proceed by induction on the structure of $\Rightarrow$.

Case G-SilentP
From G-SilentP we know there exists τ such that $\mathsf{P} \xrightarrow{\mathcal{E}\langle\tau\rangle} \mathsf{P}'$, $\pi'=\pi$. As such, from A-SilentP we have $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}', M, PB, B, \pi$. Moreover, as $\mathsf{wf}(M, PB, B, \pi)$ holds, the required result holds immediately.

Case G-Prop
From G-Prop we know there exist e and $\lambda \in \{\mathsf{B}\langle e\rangle, \mathsf{B}\langle e, -\rangle, \mathsf{P}\langle e\rangle, \mathsf{P}\langle e, -\rangle\}$ such that $\pi'=\pi.\lambda$, $\mathsf{fresh}(\lambda, \pi)$, $\mathsf{wfp}(\pi)$, $\mathsf{wfp}(\pi.\lambda)$ and $\mathsf{P}'=\mathsf{P}$. Let $\mathtt{tid}(e)=\tau$; there are seven cases to consider:

(1) $\lambda = \mathsf{B}\langle e\rangle$ for some $e \in W_{\mathsf{wb}}$; or
(2) $\lambda = \mathsf{B}\langle e\rangle$ for some $e \in SF$; or
(3) $\lambda = \mathsf{P}\langle e\rangle$ for some $e \in W_{\mathsf{nc}} \cup W_{\mathsf{wt}} \cup NTW$; or

(4) $\lambda = \mathsf{P}\langle e \rangle$ for some $e \in W_{\mathsf{wb}} \cup U_{\mathsf{wb}}$; or
(5) $\lambda = \mathsf{P}\langle e, S \rangle$ for some $e \in FL$; or
(6) $\lambda = \mathsf{B}\langle e, S \rangle$ for some $e \in FO$; or
(7) $\lambda = \mathsf{P}\langle e, w \rangle$ for some $e \in FO$.

Case (1)
Let $\mathtt{loc}(e)=x$; we then have $\mathtt{getVE}(\lambda)=e$. As $\mathsf{wfp}(\pi)$, $\mathsf{wfp}(\pi.\lambda)$ and $\mathsf{wf}(M, PB, B, \pi)$, from their definitions we know there exist b_1, b_2 such that $B(\tau)=b_1.e.b_2$. Moreover, since $\mathsf{wfp}(\pi.\lambda)$, $\mathsf{wf}(M, PB, B, \pi)$, $\mathtt{getVE}(\lambda)=e$ and $\mathtt{tid}(e)=\tau$, from Lemma 8 (part 1) we have $\forall e' \in B(\tau).\ (e', e) \notin \mathsf{PPO}(B(\tau))$ and thus $\forall e' \in b_1.\ (e', e) \notin \mathsf{PPO}(B(\tau))$. Consequently, from AM-PropW1 we have $M, PB, B \xrightarrow{\mathsf{B}\langle e \rangle} M, PB[x \mapsto PB(x).e], B[\tau \mapsto b_1.b_2]$. As such, from A-PropM we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M, PB[x \mapsto PB(x).e], B[\tau \mapsto b_1.b_2], \pi.\lambda$$

That is, there exists $M' = M$, $PB' = PB[x \mapsto PB(x).e]$ and $B' = B[\tau \mapsto b_1.b_2]$ such that $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$, as required.

Case (2)
We then have $\mathtt{getVE}(\lambda)=e$.

As $\mathsf{wfp}(\pi)$, $\mathsf{wfp}(\pi.\lambda)$ and $\mathsf{wf}(M, PB, B, \pi)$, from their definitions we know there exist b', b such that $B(\tau)=b'.e.b$. Moreover, since $\mathsf{wfp}(\pi.\lambda)$, $\mathsf{wf}(M, PB, B, \pi)$, $\mathtt{getVE}(\lambda)=e$ and $\mathtt{tid}(e)=\tau$, from Lemma 8 (part 8) we know $\forall y.\ PB(y) \cap FO_\tau = \emptyset$. Similarly, from Lemma 8 (part 1) we have $\forall e' \in B(\tau).\ (e', e) \notin \mathsf{PPO}(B(\tau))$ and thus $\forall e' \in b'.\ (e', e) \notin \mathsf{PPO}(B(\tau))$. Lastly, in what follows we show that $b'=\epsilon$ and thus $B(\tau)=e.b$. Consequently, from AM-PropSF we have $M, PB, B \xrightarrow{\mathsf{B}\langle e \rangle} M, PB, B[\tau \mapsto b]$. As such, from A-PropM we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M, PB, B[\tau \mapsto b], \pi.\lambda$$

That is, there exists $M' = M, PB' = PB$ and $B' = B[\tau \mapsto b]$ such that $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$, as required.

We next show that $b'=\epsilon$. Let us proceed by contradiction and assume there exists e' such that $e' \in b'$. As $B(\tau)=b'.e.b$, from the definition of $\mathsf{PO}(.)$ we have $(e', e) \in \mathsf{PO}(B(\tau))$. As such, since $e \in SF$ and $e' \in$ BEvent, from the definition of $\mathsf{PPO}(.)$ we also have $(e', e) \in \mathsf{PPO}(B(\tau))$. That is, $e' \in pb' \wedge (e', e) \notin \mathsf{PPO}(B(\tau))$, contradicting our earlier result, namely $\forall e' \in b'.\ (e', e) \notin \mathsf{PPO}(B(\tau))$.

Case (3)
Let $\mathtt{loc}(e)=x$; as $e \in W_{\mathtt{nc}} \cup W_{\mathtt{wt}} \cup NTW$, we then have $\mathtt{getVE}(\lambda)=e$. As $\mathsf{wfp}(\pi)$, $\mathsf{wfp}(\pi.\lambda)$ and $\mathsf{wf}(M, PB, B, \pi)$, from their definitions we know there exist b_1, b_2 such that $B(\tau)=b_1.e.b_2$. Moreover, since $\mathsf{wfp}(\pi.\lambda)$, $\mathsf{wf}(M, PB, B, \pi)$, $\mathtt{getVE}(\lambda)=e$ and $\mathtt{tid}(e)=\tau$, from Lemma 8 (part 1) we have $\forall e' \in B(\tau).\ (e', e) \notin \mathsf{PPO}(B(\tau))$ and thus $\forall e' \in b_1.\ (e', e) \notin \mathsf{PPO}(B(\tau))$. Additionally, if $x \in \mathrm{Loc}_{\mathsf{wb}} \wedge e \in NTW$, since $\mathsf{wfp}(\pi.\lambda)$, $\mathsf{wf}(M, PB, B, \pi)$, $\mathtt{getVE}(\lambda)=e$ and $\mathtt{tid}(e)=\tau$, from Lemma 8 (part 3) we have $PB(x)=\epsilon$. Consequently, from AM-PropW2 and AM-PropNTW we have $M, PB, B \xrightarrow{\mathsf{B}\langle e \rangle} M[x \mapsto e], PB, B[\tau \mapsto b_1.b_2]$. As such, from A-PropM we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M[x \mapsto e], PB, B[\tau \mapsto b_1.b_2], \pi.\lambda$$

That is, there exists $M' = M[x \mapsto e]$, $PB' = PB$ and $B' = B[\tau \mapsto b_1.b_2]$ such that $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$.

Case (4)
Let $\mathtt{loc}(e)=x$; as $e \in W_{\mathsf{wb}} \cup U_{\mathsf{wb}}$, we then have $\mathtt{getPE}(\lambda)=e$. Moreover, since $\mathtt{wfp}(\pi.\lambda)$, $\mathtt{wf}(M, PB, B, \pi)$, $\mathtt{getVE}(\lambda)=e$ and $\mathtt{tid}(e)=\tau$, from Lemma 8 (part 9) we know there exists pb such that $PB(x) = e.pb$. Consequently, from AM-PersistW we have $M, PB, B \xrightarrow{\mathsf{P}\langle e \rangle} M[x \mapsto e], PB[x \mapsto pb], B$. As such, from A-PropM we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M[x \mapsto e], PB[x \mapsto pb], B, \pi.\lambda$$

That is, there exists $M' = M[x \mapsto e]$, $PB' = PB[x \mapsto pb]$ and $B' = B$ such that $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$.

Case (5)
Let $\mathtt{loc}(e)=x$. As $\mathtt{wfp}(\pi)$, $\mathtt{wfp}(\pi.\lambda)$ and $\mathtt{wf}(M, PB, B, \pi)$, from their definitions we know there exist b_1, b_2 such that $B(\tau)=b_1.e.b_2$. Moreover, since $\mathtt{wfp}(\pi.\lambda)$, $\mathtt{wf}(M, PB, B, \pi)$, $\mathtt{getVE}(\lambda)=e$ and $\mathtt{tid}(e)=\tau$, from Lemma 8 (part 1) we have $\forall e' \in B(\tau).\ (e', e) \notin \mathsf{PPO}(B(\tau))$ and thus $\forall e' \in b_1.\ (e', e) \notin \mathsf{PPO}(B(\tau))$. Additionally, from Lemma 8 (part 6) we know $\forall y.\ (x, y) \in \mathsf{scl} \Rightarrow PB(y)=\emptyset$ and that $S = \left\{M(y) \,\middle|\, (x, y) \in \mathsf{scl}\right\}$. Consequently, from AM-PropFL we have $M, PB, B \xrightarrow{\mathsf{P}\langle e, S \rangle} M, PB, B[\tau \mapsto b_1.b_2]$. As such, from A-PropM we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M, PB, B[\tau \mapsto b_1.b_2], \pi.\lambda$$

That is, there exists $M' = M$, $PB' = PB$ and $B' = B[\tau \mapsto b_1.b_2]$ such that $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$.

The proof of case (6) is analogous to that of (5) (using Lemma 8, part 7) and is omitted here.

Case (7)
Let $\mathtt{loc}(e)=x$; we then have $\mathtt{getPE}(\lambda)=e$. Moreover, since $\mathtt{wfp}(\pi.\lambda)$, $\mathtt{wf}(M, PB, B, \pi)$, $\mathtt{getVE}(\lambda)=e$ and $\mathtt{tid}(e)=\tau$, from Lemma 8 (part 10) we know there exists pb such that $PB(x) = e.pb$ and $M(x)=w$. Consequently, from AM-PersistFO we have $M, PB, B \xrightarrow{\mathsf{P}\langle e, w \rangle} M, PB[x \mapsto pb], B$. As such, from A-PropM we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M, PB[x \mapsto pb], B, \pi.\lambda$$

That is, there exists $M' = M$, $PB' = PB[x \mapsto pb]$ and $B' = B$ such that $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$., as required.

Case G-Step
We know there exists e, r, u and $\lambda \in \{\mathsf{R}\langle r, e \rangle, \mathsf{W}\langle e \rangle, \mathsf{NTW}\langle e \rangle, \mathsf{U}\langle u, e \rangle, \mathsf{MF}\langle e \rangle, \mathsf{SF}\langle e \rangle, \mathsf{FO}\langle e \rangle, \mathsf{FL}\langle e \rangle\}$ such that $\pi'=\pi.\lambda$, $\mathtt{fresh}(\lambda, \pi)$, $\mathtt{wfp}(\pi)$, $\mathtt{wfp}(\pi.\lambda)$ and $\mathsf{P} \xrightarrow{\lambda} \mathsf{P}'$. There are now eight cases to consider:
(1) $\lambda = \mathsf{R}\langle r, e \rangle$
(2) $\lambda = \mathsf{W}\langle e \rangle$
(3) $\lambda = \mathsf{NTW}\langle e \rangle$
(4) $\lambda = \mathsf{U}\langle u, e \rangle$
(5) $\lambda = \mathsf{MF}\langle e \rangle$
(6) $\lambda = \mathsf{SF}\langle e \rangle$
(7) $\lambda = \mathsf{FO}\langle e \rangle$
(8) $\lambda = \mathsf{FL}\langle e \rangle$

Case (1): $\lambda = \mathsf{R}\langle r, e \rangle$
Let $\mathtt{tid}(r)=\tau$ and $\mathtt{loc}(r)=x$. There are then two cases to consider: i) $x \in \mathrm{Loc}_{\mathsf{c}}$; or ii) $x \in \mathrm{Loc}_{\mathsf{nc}}$.

In case (i), let $PB(x){=}pb$ and $B(\tau){=}b$. As $\mathtt{wfp}(\pi.\lambda)$, $\mathtt{wf}(M, PB, B, \pi)$, $\lambda = \mathsf{R}\langle r, e\rangle$ and $\mathtt{tid}(r){=}\tau$, from Lemma 8 (part 4) we know $\mathtt{rd}(M, pb, b, x) = e$. From AM-ReadC we then have $M, PB, B \xrightarrow{\mathsf{R}\langle r,e\rangle} M, PB, B$. As such, from A-Step we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M, PB, B, \pi.\lambda$$

That is, there exists $M'{=}M$, $PB'{=}PB$, $B'{=}B$ such that $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$, as required.

The proof of case (ii) is analogous to that of part (i) (using Lemma 8, part 5 instead of part 4) and is omitted.

Case (2)*:* $\lambda = \mathsf{W}\langle e\rangle$
Let $\mathtt{tid}(e){=}\tau$. From AM-Write we then have $M, PB, B \xrightarrow{\mathsf{W}\langle e\rangle} M, PB, B[\tau \mapsto B(\tau).e]$. As such, from A-Step we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M, PB, B[\tau \mapsto B(\tau).e], \pi.\lambda$$

That is, there exists $M'{=}M$, $PB'{=}PB$ and $B'{=}B[\tau \mapsto B(\tau).e]$ such that $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$, as required.

Case (4)*:* $\lambda = \mathsf{U}\langle u, e\rangle$
We then have $\mathtt{getVE}(\lambda){=}u \in U$. Let $\mathtt{tid}(u){=}\tau$ and $\mathtt{loc}(r){=}x$. There are then two cases to consider: i) $x \in \mathrm{Loc}_{\mathsf{wb}}$; or ii) $x \notin \mathrm{Loc}_{\mathsf{wb}}$.

In case (i), let $PB(x){=}pb$ and $B(\tau){=}b$. As $\mathtt{wfp}(\pi.\lambda)$, $\mathtt{wf}(M, PB, B, \pi)$, $\lambda = \mathsf{U}\langle u, e\rangle$, $\mathtt{tid}(u){=}\tau$ and $\mathtt{getVE}(\lambda){=}u \in U$, from Lemma 8 (part 2) we know $b{=}\epsilon$. Analogously, from Lemma 8 (part 8) we know $\forall y.\ PB(y) \cap FO_\tau = \emptyset$. Similarly, from Lemma 8 (part 4) we know $\mathtt{rd}(M, pb, b, x) = e$. From AM-RMW1 we then have $M, PB, B \xrightarrow{\mathsf{U}\langle u,e\rangle} M, PB[x \mapsto pb.u], B$. As such, from A-Step we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M, PB[x \mapsto pb.u], B, \pi.\lambda$$

That is, $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$, where $M'{=}M$, $PB'{=}PB[x \mapsto pb.u]$ and $B'{=}B$, as required.

The proof of case (ii) is analogous to that of case (i) (using Lemma 8, part 5 instead of part 4) and is omitted.

Case (5)*:* $\lambda = \mathsf{MF}\langle e\rangle$
We then have $\mathtt{getVE}(\lambda){=}e \in MF$. Let $\mathtt{tid}(e){=}\tau$ and $B(\tau){=}b$. As $\mathtt{wfp}(\pi.\lambda)$, $\mathtt{wf}(M, PB, B, \pi)$, $\lambda = \mathsf{MF}\langle e\rangle$, $\mathtt{tid}(e){=}\tau$ and $\mathtt{getVE}(\lambda){=}e \in MF$, from Lemma 8 (part 2) we know $b{=}\epsilon$. Analogously, from Lemma 8 (part 8) we know $\forall y.\ PB(y) \cap FO_\tau = \emptyset$. From AM-MF we then have $M, PB, B \xrightarrow{\mathsf{MF}\langle e\rangle} M, PB, B$. As such, from A-Step we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M, PBB, \pi.\lambda$$

That is, $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$, where $M'{=}M$, $PB'{=}PB$ and $B'{=}B$, as required.

Case (6)*:* $\lambda = \mathsf{SF}\langle e\rangle$
Let $\mathtt{tid}(e){=}\tau$. From AM-SF we then have $M, PB, B \xrightarrow{\mathsf{SF}\langle e\rangle} M, PB, B[\tau \mapsto B(\tau).e]$. As such, from A-Step we have:

$$\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M, PB, B[\tau \mapsto B(\tau).e], \pi.\lambda$$

That is, there exists $M'{=}M$, $PB'{=}PB$ and $B'{=}B[\tau \mapsto B(\tau).e]$ such that $\mathsf{P}, M, PB, B, \pi \Rightarrow \mathsf{P}, M', PB', B', \pi'$, as required.

The proofs of case (7) and case (8) are analogous to that of (6) and thus omitted here. □

Corollary 1. *For all* $\mathsf{P}, \pi, \mathsf{P}', M, PB, B$, *if* $\mathsf{P}, \epsilon \Rightarrow^* \mathsf{P}', \pi$, *then there exists* (M, PB, B) *such that:*

- $\mathsf{P}, M_0, PB_0, B_0, \epsilon \Rightarrow^* \mathsf{P}', M, PB, B, \pi$
- $\mathsf{wf}(M, PB, B, \pi)$

Proof. As from the definition of well-formedness we simply have $\mathsf{wf}(M_0, PB_0, B_0, \epsilon)$, the first result follows from Lemma 9 and induction on the length of $\Rightarrow^*$. The second result then follows from the first result and Lemma 1. □

Lemma 10. *For all PEx86-consistent executions* G, *and all* π, M, *if* $\pi = \mathtt{getPath}(G)$ *and* $\mathsf{wf}(M, -, -, \pi)$, *then* $M = G.P$.

Proof. Pick an PEx86-consistent execution $G = (E, P, \mathsf{po}, \mathsf{rf}, \mathsf{mo}, \mathsf{pf})$ and π, M such that $\pi = \mathtt{getPath}(G)$ and $\mathsf{wf}(M, -, -, \pi)$. As $\pi = \mathtt{getPath}(G)$, from Lemma 6 we then know that $\mathsf{wfp}(\pi)$ holds. It then suffices to show that for all $x \in \mathrm{Loc}$, $M(x) = G.P(x)$.

Pick an arbitrary $x \in \mathrm{Loc}$. Let $M(x) = e$. As $\mathsf{wf}(M, -, -, \pi)$, we know $M(x) = \mathtt{pread}(\pi, x)$ and thus $e \in ST_x$ and there exist π_1, π_2, λ such that $\pi = \pi_1.\lambda.\pi_2$, $S = \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \mathtt{getPE}(\lambda') = e'\right\} = \emptyset$, and $\mathtt{getPE}(\lambda) = e$. There are now two cases to consider: 1) $x \in \mathrm{Loc}_{\mathtt{nc}} \cup \mathrm{Loc}_{\mathtt{wt}}$; or 2) $x \in \mathrm{Loc}_{\mathtt{wb}}$.

In case (1), it suffices to show that $e = \max(\mathsf{mo}_x)$. Let us proceed by contradiction and assume there exists $e' \in ST_x$ such that $(e, e') \in \mathsf{mo}_x$. From the definitions of $\mathtt{getPE}(.)$ and $\mathtt{getVE}(.)$ and since $x \notin \mathrm{Loc}_{\mathtt{wb}}$, we know that for all $e' \in ST_x$ and all λ': $\mathtt{getVE}(\lambda') = e' \Leftrightarrow \mathtt{getPE}(\lambda') = e'$. As such, we also have $S' = \left\{\lambda' \in \pi_2 \,\middle|\, \exists e' \in ST_x.\ \mathtt{getVE}(\lambda') = e'\right\} = \emptyset$, and that $\mathtt{getVE}(\lambda) = \mathtt{getPE}(\lambda) = e$, and thus from Prop. 3 we know $\mathtt{genVL}(e, G) = \mathtt{genPL}(e, G) = \lambda$. As $(e, e') \in \mathsf{mo}_x$ and G is PEx86-consistent, we know $(e, e') \in \mathsf{ob}$ and thus from Prop. 3 we know there exists λ' such that $\lambda' = \mathtt{genVL}(e', G)$ and $\mathtt{genVL}(e, G) \prec_\pi \lambda'$; i.e. (from Prop. 3) we know $\lambda \prec_\pi \lambda'$. That is, as $\pi = \pi_1.\lambda.\pi_2$, we know $\lambda' \in \pi_2$. Moreover, as $\lambda' = \mathtt{genVL}(e', G)$, from Prop. 3 we have $\mathtt{getVE}(\lambda') = e'$. Consequently, we know $\lambda' \in \pi_2$, $\mathtt{getVE}(\lambda') = e'$ and $e' \in ST_x$, and thus $\lambda' \in S'$. This, however, contradicts our earlier result that $S' = \emptyset$.

In case (2), let us proceed by contradiction and assume $P(x) = w$ and $w \neq e$. As $\pi = \mathtt{getPath}(G)$ and $\mathtt{getPE}(\lambda) = e$, from Prop. 3 we know $\mathtt{genPL}(e, G) = \lambda$. Moreover, as $x \in \mathrm{Loc}_{\mathtt{wb}}$, from the construction of π ($\pi = \mathtt{getPath}(G)$) we know that either $e \in NTW$ or $e \in \mathcal{PW}$. If $e \in NTW$, since $w \neq x$ and G is consistent, from weak-persist we know $(e, w) \in \mathsf{mo}$ and $\mathtt{getPE}(\lambda_w) = w$, i.e. $\mathtt{genPL}(w, G) \in \pi$. On the other hand, if $e \in \mathcal{PW}$, since $w \neq x$ from the definition of $\mathcal{PW}$ we know $(e, w) \in \mathsf{mo}$ and $w \in \mathcal{PW}$; moreover, from the construction of π we know $\mathtt{genPL}(w, G) \in \pi$. That is, in both cases we have $(e, w) \in \mathsf{mo}$ and that there exists $\lambda^p_w \in \pi$ such that $\lambda^p_w = \mathtt{genPL}(w, G)$ and thus (from Prop. 3) $\mathtt{getPE}(\lambda^p_w) = w$. As such, since $\mathsf{mo} \subseteq \mathsf{ob}$, from Prop. 3 we know there exist λ_w, λ_e such that $\mathtt{genVL}(e, G) = \lambda_e$, $\mathtt{getVE}(\lambda_e) = e$, $\mathtt{genVL}(w, G) = \lambda_w$, $\mathtt{getVE}(\lambda_w) = w$ and $\lambda_e \prec_\pi \lambda_w$. Moreover, as $\lambda_e \prec_\pi \lambda_w$, $\mathtt{getVE}(\lambda_e) = e$, $\mathtt{getVE}(\lambda_w) = w$, $\mathtt{getPE}(\lambda) = e$, $\mathtt{getPE}(\lambda^p_w) = w$, $w, e \in ST_x$, and $\mathsf{wfp}(\pi)$ holds, we know $\lambda \prec_\pi \lambda^p_w$. As $\pi = \pi_1.\lambda.\pi_2$, we thus have $\lambda^p_w \in \pi_2$. That is, $w \in ST_x$, $\lambda^p_w \in \pi_2$ and $\mathtt{getPE}(\lambda^p_w) = w$, and thus $\lambda^p_w \in S$. This, however, contradicts our assumption that $S = \emptyset$. □

Theorem 5 (Completeness). *For all programs* P *and all PEx86-consistent executions* G *of* P, *there exist* M, PB, B, π *such that:*

(1) $\mathsf{P}, M_0, PB_0, B_0, \epsilon \Rightarrow^* \mathsf{P}', M, PB, B, \pi$
(2) $M = G.P$

Proof. Pick an arbitrary program P and an PEx86-consistent executions G of P. Let $\pi \triangleq \mathtt{getPath}(G)$. From Lemma 7 we then know there exists P' such that $\mathsf{P}, \epsilon \Rightarrow^* \mathsf{P}', \pi$. Consequently, for part 1 from Corollary 1 we know there exists M, PB, B such that $\mathsf{P}, M_0, PB_0, B_0, \epsilon \Rightarrow^* \mathsf{P}', M, PB, B, \pi$ and $\mathsf{wf}(M, PB, B, \pi)$, as required. For part 2, as $\mathsf{wf}(M, PB, B, \pi)$ holds, from Lemma 10 we have $M = G.P$, as required. □

B.2 Equivalence of PEx86 Operational and Event-Annotated Semantics

Let

$$R_l \triangleq \left\{ ((\tau : l), \lambda) \;\middle|\; \begin{array}{l} \texttt{tid}(\lambda){=}\tau \wedge \exists e, x. \\ \quad \left(\begin{array}{l} (\texttt{getE}(\lambda){=}e \wedge \texttt{lab}(e){=}l) \\ \vee (\lambda \in \left\{ \mathcal{E}\langle\tau\rangle, \mathsf{B}\langle-\rangle, \mathsf{P}\langle-\rangle, \mathsf{B}\langle-,-\rangle, \mathsf{P}\langle-,-\rangle \right\} \wedge l = \epsilon) \end{array} \right) \end{array} \right\}$$

Lemma 11. *For all* P, P'*:*

- *for all* τ, l*, if* $\mathsf{P}, \xrightarrow{\tau:l} \mathsf{P}'$*, then there exists* λ *such that:* $((\tau, l), \lambda) \in R_l$ *and* $\mathsf{P} \xrightarrow{\lambda} \mathsf{P}'$
- *for all* λ*, if* $\mathsf{P} \xrightarrow{\lambda} \mathsf{P}'$*, then there exists* τ, l *such that:* $((\tau, l), \lambda) \in R_l$ *and* $\mathsf{P} \xrightarrow{\tau:l} \mathsf{P}'$

Proof. By straightforward induction on the structures of $\xrightarrow{\tau:l}$ and $\xrightarrow{\lambda}$. □

Let

$$R_m \triangleq \left\{ \begin{array}{c} ((\mathsf{M}, \mathsf{PB}, \mathsf{B}), \\ (M, PB, B)) \end{array} \;\middle|\; \begin{array}{l} (\mathsf{M}, \rightharpoonup, \mathsf{B}) \in \textsc{Mem} \times \textsc{PBMap} \times \textsc{BMap} \\ \wedge\ (M, PB, B) \in \textsc{AMem} \times \textsc{APBMap} \times \textsc{ABMap} \\ \wedge\ \forall x, v.\ \mathsf{M}(x) = v \Leftrightarrow \texttt{val}_{\mathtt{w}}(M(x)) = v \\ \wedge\ \mathsf{sim}_{\mathsf{pb}}(\mathsf{PB}, PB) \wedge \mathsf{sim}_{\mathsf{b}}(\mathsf{B}, B) \end{array} \right\}$$

$$\mathsf{sim}_{\mathsf{b}}(\mathsf{PB}, PB) \overset{\text{def}}{\Leftrightarrow} dom(\mathsf{PB}){=}dom(PB) \wedge \forall x \in dom(\mathsf{PB}).\ \mathsf{sim}_{\mathsf{pb}}(\mathsf{PB}(x), PB(x))$$

$$\begin{array}{rl} \mathsf{sim}_{\mathsf{pb}}(\mathsf{pb}, pb) \overset{\text{def}}{\Leftrightarrow} & \mathsf{pb} = pb = \epsilon \\ & \vee\ \exists \mathsf{pb}', pb', v, e.\ \mathsf{pb}{=}\mathsf{w}(v).\mathsf{pb}' \wedge pb{=}e.pb' \wedge \texttt{val}_{\mathtt{w}}(e){=}v \wedge \mathsf{sim}_{\mathsf{pb}}(\mathsf{pb}', pb') \\ & \vee\ \exists \mathsf{pb}', pb', \tau, e.\ \mathsf{pb}{=}\mathsf{fo}(\tau).\mathsf{pb}' \wedge pb{=}e.pb' \wedge e \in FO_\tau \wedge \mathsf{sim}_{\mathsf{pb}}(\mathsf{pb}', pb') \end{array}$$

$$\mathsf{sim}_{\mathsf{b}}(\mathsf{B}, B) \overset{\text{def}}{\Leftrightarrow} dom(\mathsf{B}){=}dom(B) \wedge \forall \tau \in dom(\mathsf{B}).\ \mathsf{sim}_{\mathsf{b}}(\mathsf{B}(\tau), B(\tau))$$

$$\begin{array}{rl} \mathsf{sim}_{\mathsf{b}}(\mathsf{b}, b) \overset{\text{def}}{\Leftrightarrow} & (\mathsf{b}{=}b{=}\epsilon) \\ & \vee\ \exists \mathsf{b}', b', l, e.\ \mathsf{b} = l.\mathsf{b}' \wedge b = e.b' \wedge \texttt{lab}(e){=}l \wedge \mathsf{sim}_{\mathsf{b}}(\mathsf{b}', b') \end{array}$$

Lemma 12. *Let* $\mathsf{PB}_0 \triangleq \lambda x.\epsilon$ *and* $\mathsf{B}_0 \triangleq \lambda\tau.\epsilon$*. For all* $M, PB, B, \mathsf{M}, \mathsf{PB}, \mathsf{B}$*:*

- $((\mathsf{M}_0, \mathsf{PB}_0, \mathsf{B}_0), (M_0, PB_0, B_0)) \in R_m$
- *for all* $\mathsf{M}', \mathsf{PB}', \mathsf{B}', \tau, l$ *such that* $(\mathsf{M}, \mathsf{PB}, \mathsf{B}) \xrightarrow{\tau:l} (\mathsf{M}', \mathsf{PB}', \mathsf{B}')$*:*
  if $((\mathsf{M}, \mathsf{PB}, \mathsf{B}), (M, PB, B)) \in R_m$
  then there exist M', PB', B', λ *such that* $((\tau, l), \lambda) \in R_l$*,* $((\mathsf{M}', \mathsf{PB}', \mathsf{B}'), (M', PB', B')) \in R_m$ *and* $(M, PB, B) \xrightarrow{\lambda} (M', PB', B')$
- *for all* M', PB', B', λ *such that* $(M, PB, B) \xrightarrow{\lambda} (M', PB', B')$*:*
  if $((\mathsf{M}, \mathsf{PB}, \mathsf{B}), (M, PB, B)) \in R_m$
  then there exist $\mathsf{M}', \mathsf{PB}', \mathsf{B}', \tau, l$ *such that* $((\tau, l), \lambda) \in R_l$*,* $((\mathsf{M}', \mathsf{PB}', \mathsf{B}'), (M', PB', B')) \in R_m$ *and* $(\mathsf{M}, \mathsf{PB}, \mathsf{B}) \xrightarrow{\tau:l} (\mathsf{M}', \mathsf{PB}', \mathsf{B}')$

Proof. The first part follows immediately from the definitions of M_0, PB_0, B_0, M_0, PB_0, B_0. The last two parts follow from straightforward induction on the structures of $\xrightarrow{\tau:l}$ and $\xrightarrow{\lambda}$. □

Let

$$R \triangleq \left\{ \begin{array}{l} ((\mathsf{P}, \mathsf{M}, \mathsf{PB}, \mathsf{B}), \\ (\mathsf{P}, M, PB, B, \pi)) \end{array} \;\middle|\; \mathsf{P} \in \textsc{Prog} \wedge \pi \in \textsc{Path} \wedge ((\mathsf{M}, \mathsf{PB}, \mathsf{B}), (M, PB, B)) \in R_m \right\}$$

Lemma 13. *For all* $\mathsf{P}, \mathsf{M}, \mathsf{PB}, \mathsf{B}, M, PB, B, M', PB', B', \pi$*:*

- $((\mathsf{P}, \mathsf{M}_0, \mathsf{PB}_0, \mathsf{B}_0), (\mathsf{P}, M_0, PB_0, B_0, \epsilon)) \in R$
- *for all* $\mathsf{P}', \mathsf{M}', \mathsf{PB}', \mathsf{B}'$ *such that* $(\mathsf{P}, \mathsf{M}, \mathsf{PB}, \mathsf{B}) \Rightarrow (\mathsf{P}', \mathsf{M}', \mathsf{PB}', \mathsf{B}')$*:*
  if $((\mathsf{P}, \mathsf{M}, \mathsf{PB}, \mathsf{B}), (\mathsf{P}, M, PB, B, \pi)) \in R$
  then there exist M', PB', B', π' *such that* $((\mathsf{P}', \mathsf{M}', \mathsf{PB}', \mathsf{B}'), (\mathsf{P}', M', PB', B', \pi')) \in R$ *and* $(\mathsf{P}, M, PB, B, \pi) \Rightarrow (\mathsf{P}', M', PB', B', \pi')$.
- *for all* $\mathsf{P}', M', PB', B', \pi'$ *such that* $(\mathsf{P}, M, PB, B, \pi) \Rightarrow (\mathsf{P}', M', PB', B', \pi')$*:*
  if $((\mathsf{P}, \mathsf{M}, \mathsf{PB}, \mathsf{B}), (\mathsf{P}, M, PB, B, \pi)) \in R$
  then there exist $\mathsf{M}', \mathsf{PB}', \mathsf{B}'$ *such that* $((\mathsf{P}', \mathsf{M}', \mathsf{PB}', \mathsf{B}') , (\mathsf{P}', M', PB', B', \pi')) \in R$ *and* $(\mathsf{P}, \mathsf{M}, \mathsf{PB}, \mathsf{B}) \Rightarrow (\mathsf{P}', M', PB', B')$.

Proof. The proof of the first part follows immediately from the definition of R and Lemma 12. The proofs of the last two parts follow from straightforward induction on the structures of $\xrightarrow{\tau:l}$, $\xrightarrow{\lambda}$, Lemma 11 and Lemma 12. □

Theorem 6 (Intermediate and operational semantics equivalence)**.** *For all* P*:*

- *for all* $\mathsf{P}', \mathsf{M}, \mathsf{PB}, \mathsf{B}$*:*
  if $\mathsf{P}, \mathsf{M}_0, \mathsf{PB}_0, \mathsf{B}_0 \Rightarrow^* \mathsf{P}', \mathsf{M}, \mathsf{PB}, \mathsf{B}$
  then there exist M, PB, B, π *such that* $\mathsf{P}, M_0, PB_0, B_0, \epsilon \Rightarrow^* \mathsf{P}', M, PB, B, \pi$ *and* $((\mathsf{M}, \mathsf{PB}, \mathsf{B}), (M, PB, B)) \in R_m$
- *for all* $\mathsf{P}', M, PB, B, \pi$*:*
  if $\mathsf{P}, M_0, PB_0, B_0, \epsilon \Rightarrow^* \mathsf{P}', M, PB, B, \pi$
  then there exists $\mathsf{M}, \mathsf{PB}, \mathsf{B}$ *such that* $\mathsf{P}, \mathsf{M}_0, \mathsf{PB}_0, \mathsf{B}_0 \Rightarrow^* \mathsf{P}', \mathsf{M}, \mathsf{PB}, \mathsf{B}$ *and* $((\mathsf{M}, \mathsf{PB}, \mathsf{B}), (M, PB, B)) \in R_m$.

Proof. Follows from Lemma 13 and straightforward induction on the length of $\Rightarrow^*$. □